THE HOMOTOPY THEORY OF
DIFFERENTIABLE SHEAVES

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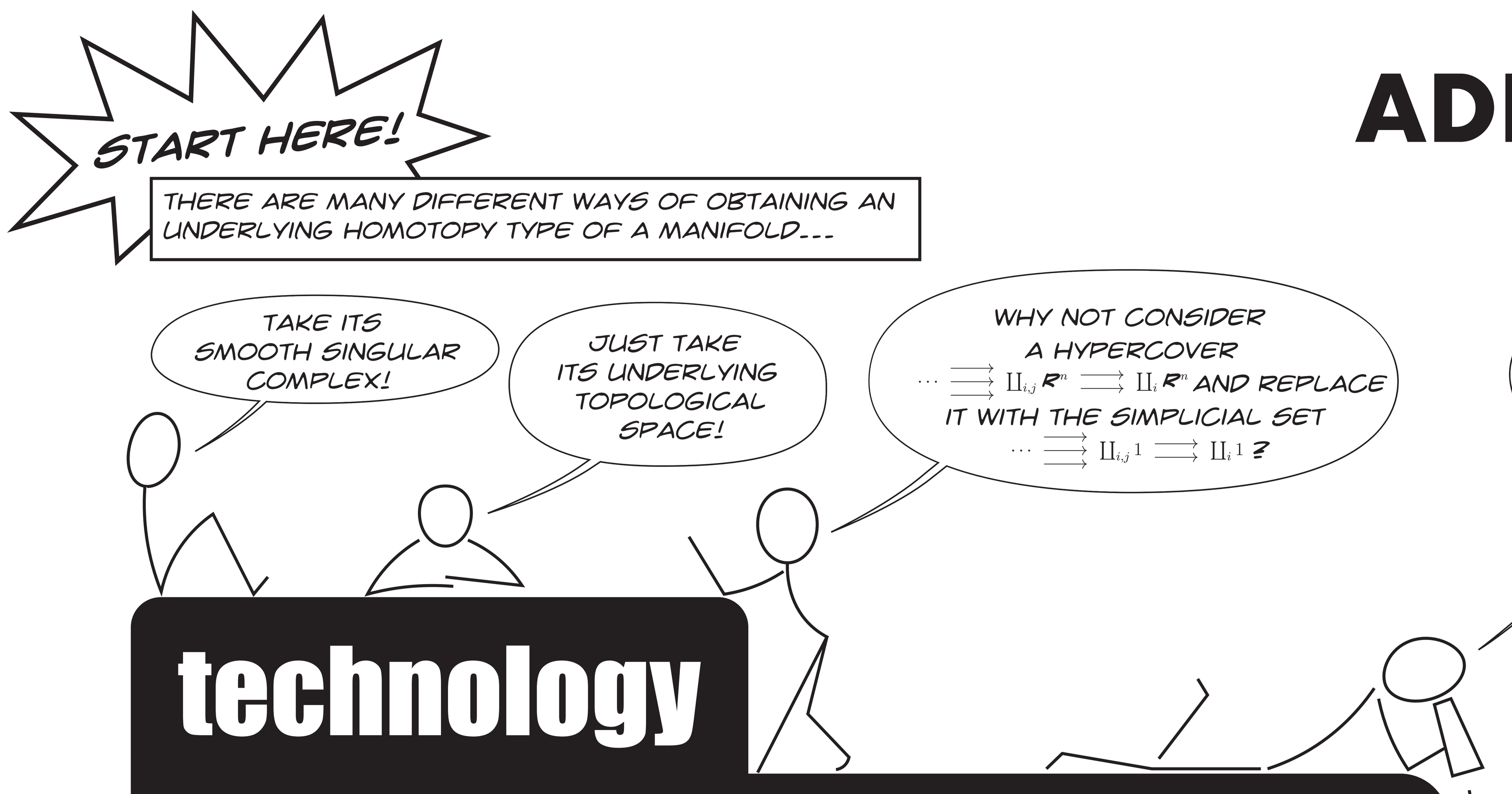
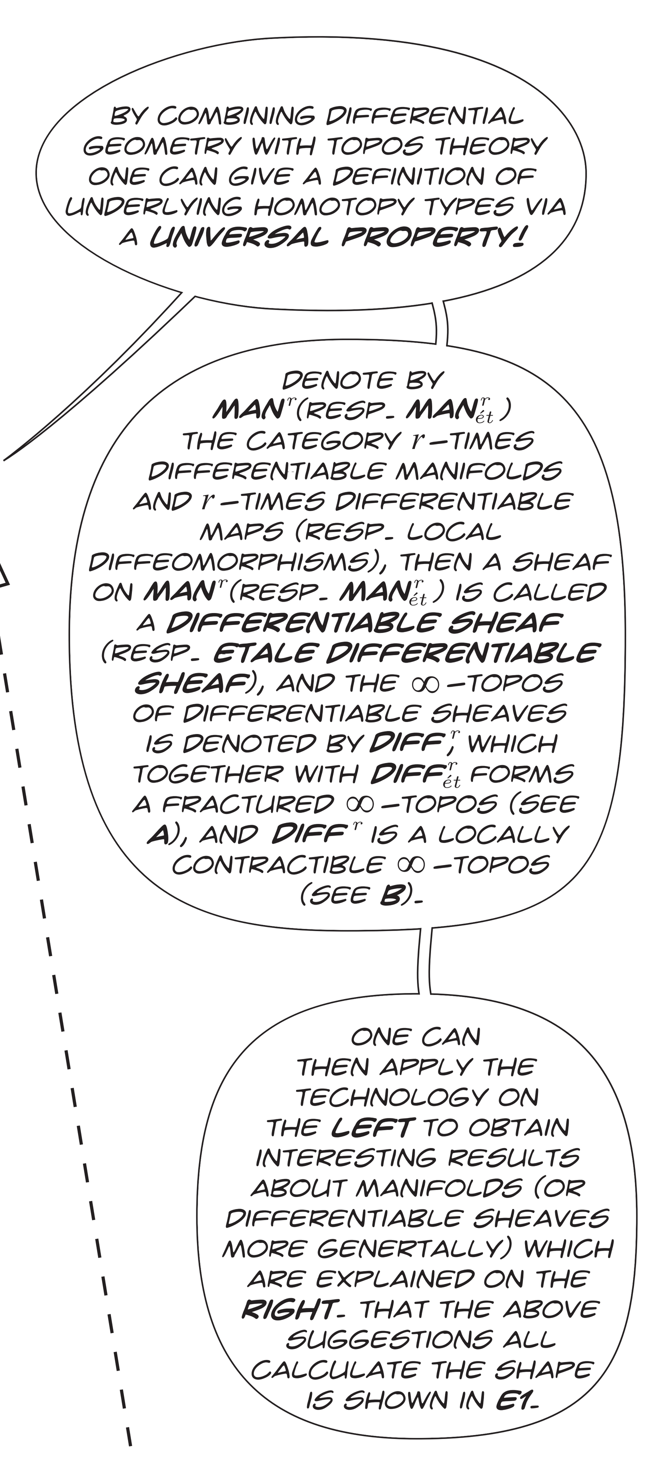




technology

A: Fractured ∞-toposes

A **fractured ∞-topos** is an adjunction $j_! : \mathcal{E}^{\mathrm{corp}} \rightleftarrows \mathcal{E} : j^*$ between ∞-toposes $\mathcal{E}^{\mathrm{corp}}$ and $\mathcal{E}$ such that (among other things), $j_!$ preserves pullbacks, $\mathcal{E}$ is generated under colimits by the image of $j_!$, and j^* admits a right adjoint (see [Lur18, Def. 20.1.2.1]).

Given a geometry $(G, G^{\mathrm{ad}}, \tau)$ (see [Lur09, Def. 1.2.5]), taking sheaves on (G, τ) and $(G^{\mathrm{ad}}, \tau|_{G^{\mathrm{ad}}})$ will respectively produce $\mathcal{E}$ and $\mathcal{E}^{\mathrm{corp}}$ of a fractured ∞-topos, and the restriction functor gives j^*. The theory of fractured ∞-toposes may then be viewed as a "coordinate free" version of the theory of geometries developed in [Lur09].

Taking G to be the opposite ∞-category of compact commutative ring spectra, G^{ad} to be the wide subcategory where the morphisms are the étale morphisms, and τ to be the étale topology on G, then, surprisingly, $\mathcal{E}^{\mathrm{corp}}$ turns out to be equivalent to the ∞-category of Deligne-Mumford stacks (satisfying a certain finiteness condition) and étale morphisms between them. This pattern generalises to other geometries, so we see that fractured toposes may also be seen as a way to formalise the relationship between petit and gros ∞-toposes of Deligne-Mumford stacks.

B: Shapes and locally contractible ∞-toposes

For any ∞-topos $\mathcal{E}$ the constant sheaf functor $\mathcal{E} \leftarrow \mathcal{S} : \pi^*$ admits a pro-left adjoint $\pi_! : \mathcal{E} \to \mathrm{Pro}(\mathcal{S})$. For any object $E \in \mathcal{E}$ the pro-homotopy type $\pi_! E$ is called the **shape** of E. We write $\pi_! \mathcal{E} := \pi_! 1$. Any geometric morphism $f : \mathcal{E} \to \mathcal{F}$ induces a map $\pi_! \mathcal{E} \to \pi_! \mathcal{F}$. Moreover, $\pi_! E = \pi_! \mathcal{E}_{/E}$ for any E in $\mathcal{E}$.

An ∞-topos $\mathcal{E}$ is called **locally contractible** if $\pi_!$ factors through the inclusion $\mathcal{S} \hookrightarrow \mathrm{Pro}(\mathcal{S})$, and thus constitutes a true left adjoint to π^*.

E.g., for any small ∞-category A the diagonal functor $\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}) \leftarrow \mathcal{S}$ admits both a left and a right adjoint given by taking colimits and limits respectively, so that $\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S})$ is a locally contractible ∞-topos; moreover, we have $\pi_! \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}) = \mathrm{colim}\, 1 = BA$. A functor $A \to B$ between small ∞-categories is initial (a.k.a. coinitial, a.k.a. cofinal, a.k.a...) iff the induced pullback functor $\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}) \leftarrow \mathrm{Fun}(B^{\mathrm{op}}, \mathcal{S})$ preserves shapes.

B1: If $\mathcal{E}$ is generated by a subcategory B, all of whose objects have contractible shape, then $\mathcal{E}$ is locally contractible and both components of the adjunction $a^* : \mathrm{Fun}(B^{\mathrm{op}}, \mathcal{S}) \rightleftarrows \mathcal{E} : a_*$ preserve shapes. If we moreover have an initial functor $f : A \to B$, then the shape of any object E in $\mathcal{E}$ may be calculated as $\mathrm{colim}\, \mathcal{E}(f, E)$.

B2: If the pushforward component of a geometric morphism $\mathcal{E} \to \mathcal{F}$ admits an extra right adjoint, then the induced map $\pi_! \mathcal{E} \to \pi_! \mathcal{F}$ is an isomorphism. An adjunction between small ∞-categories A, B will give rise to such a geometric morphism, and in this case we recover the statement that $BA \simeq BB$. If $\mathcal{E}$ is a fractured ∞-topos, then for any E in $\mathcal{E}^{\mathrm{corp}}$ the induced adjunction $(j_!)_{/E} : \mathcal{E}^{\mathrm{corp}}_{/E} \rightleftarrows \mathcal{E}_{/j_! E} : (j^*)_{/E}$ is a geometric morphism where $(j^*)_{/E}$ admits an extra right adjoint, so $\pi_! E = \pi_! \mathcal{E}^{\mathrm{corp}}_{/E} = \pi_! \mathcal{E}_{/j_! E} = \pi_! j_! E$.

C: Homotopical calculi

Let $\mathcal{E}$ be locally contractible ∞-topos then, being a left adjoint, $\pi_! : \mathcal{E} \to \mathcal{S}$ commutes with colimits. For some applications, e.g., the ones described in **F**, it is useful to commute certain limits past $\pi_!$. Denote any morphism in $\mathcal{E}$ sent to an isomorphism by $\pi_!$ as a **shape equivalence**. For any E in $\mathcal{E}$ the induced map $(\pi_!)_{/E} : \mathcal{E}_{/E} \to \mathcal{S}_{/\pi_! E}$ admits a fully faithful right adjoint, so that $(\pi_!)_{/E}$ is then a localisation along the shape equivalences, and limits commuting past $\pi_!$ may be viewed as homotopy limits. Thus, one is led to construct homotopical calculi on $\mathcal{E}$.

We shall consider the case of homotopy pullbacks. A morphism $f : E \to E'$ in $\mathcal{E}$ is called **sharp** if the induced functor $\mathcal{E}_{/E} \leftarrow \mathcal{E}_{/E'} : f^*$ preserves shape equivalences. In this case, f induces a functor $\mathcal{S}_{/E} \leftarrow \mathcal{S}_{/E'}$ which is right adjoint to the postcomposition functor $\mathcal{S}_{/E} \to \mathcal{S}_{/E'}$, and thus pullbacks along f are homotopy pullbacks.

E1: The notions of model structure and fibration structure (a variant of model structure which only formalises weak equivalences and fibrations) make sense on ∞-categories, and the fibrations of any fibration structure (and thus model structure) on $\mathcal{E}$ are sharp. In the situation of **B1**, if $\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S})$ admits a cofibrantly generated model structure, e.g., when A is a test category, then it is fairly straightforward to transfer this model structure to one on $\mathcal{E}$, which is Quillen equivalent to the one on $\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S})$, where the weak equivalences are the shape equivalences, and thus yielding a supply of sharp maps on $\mathcal{E}$.

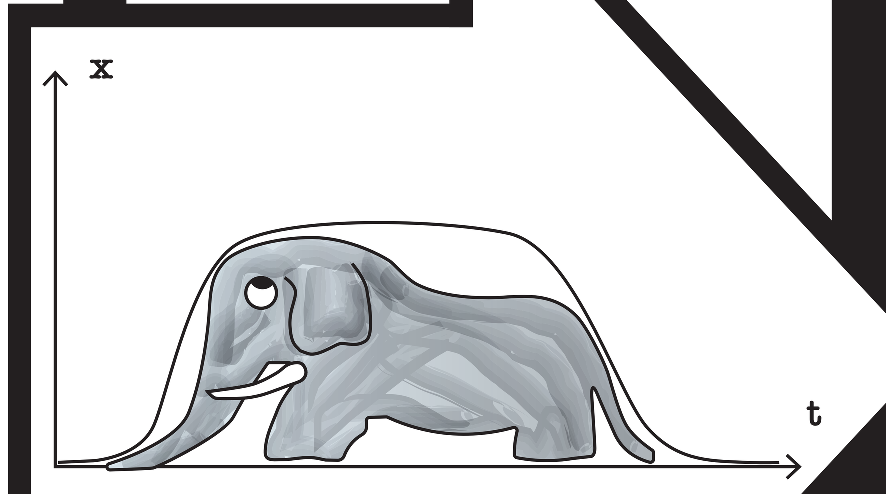


Fig: Combining differential geometry and topos theory

differentiable sheaves

D: The fractured ∞-topos of differentiable sheaves

For any r-times differentiable manifold M the ∞-topos $(\mathbf{Diff}^r_{\text{ét}})_{/M}$ is equivalent to the ∞-category of ($\mathcal{S}$-valued) sheaves on the underlying topological space of M. This allows us to relate properties of M to properties of its underlying topological space in a systematic manner.

For instance, we can show that if M is closed, then M is compact in the categorical sense in $\mathbf{Diff}^r$, i.e., $\mathbf{Diff}^r(M, \_) : \mathbf{Diff}^r \to \mathcal{S}$ preserves filtered colimits. To see this, let A be a small filtered ∞-category, then for any functor $X : A \to \mathbf{Diff}^r$ we have by [Lur09, Th. 7.3.1.16 & Rmk. 7.3.1.5] that

$$\begin{aligned}\mathbf{Diff}^r(j_! M, \mathrm{colim}_{\alpha \in A} X_\alpha) &= \mathbf{Diff}^r_{\text{ét}}(M, j^* \mathrm{colim}_{\alpha \in A} X_\alpha) = \mathbf{Diff}^r_{\text{ét}}(M, \mathrm{colim}_{\alpha \in A} j^* X_\alpha)\\ &= \mathrm{colim}_{\alpha \in A} \mathbf{Diff}^r_{\text{ét}}(M, j^* X_\alpha) = \mathrm{colim}_{\alpha \in A} \mathbf{Diff}^r(j_! M, X_\alpha).\end{aligned}$$

Surprisingly, if M is compact with non-empty boundary, then M is **not** categorically compact, as infinitely many maps $\mathbf{R}^d \to M$ are then required to specify the smooth structure on M.

E: Shapes, cofinality and differentiable sheaves

The fractured ∞-topos structure $j_! : \mathbf{Diff}^r_{\text{ét}} \to \mathbf{Diff}^r$ may be used to show that $\mathbf{Diff}^r$ is locally contractible, by showing that $\mathbf{R}^d$ has a trivial shape as an object in $\mathbf{Diff}^r_{\text{ét}}$ (and thus in $\mathbf{Diff}^r$) for all $d \geq 0$. Finite products of ∞-toposes with trivial shape again have trivial shape, so that one may reduce to the case of $d = 1$, where the statement follows from the fact that $\mathbf{R}$ is connected and that all non-zero cohomology vanish, which itself can be shown using a Galois theoretic argument.

E1: The functor $\mathbf{A}^\bullet : \Delta \to \mathbf{Cart}^r$, $[n] \mapsto \left\{ (x_0, \ldots, x_n) \in \mathbf{R}^{n+1} \,\middle|\, x_0 + \cdots + x_n = 1 \right\}$ is initial, so that by **B1** the smooth total singular complex (w.r.t. extended simplices) calculates the shape of any differentiable sheaf. These arguments can easily be modified to show that a wide array of total singular complexes (e.g. with respect to standard simplices) calculate shapes.

For any $0 \leq r \leq s \leq \infty$ the forgetful functor $u : \mathbf{Mfd}^s \to \mathbf{Mfd}^r$ is both cover preserving and reflecting, yielding a geometric morphism $u : \mathbf{Diff}^s \to \mathbf{Diff}^r$ where u^* admits a left adjoint $u_!$. The unique geometric morphism $\pi_s : \mathbf{Diff}^s \to \mathcal{S}$ is given by $\pi_r \circ u$, so that $(\pi_s)_! = (\pi_r)_! \circ u_!$; setting $r = 0$ and $s = \infty$, we see that the shape of any smooth manifold is modelled by its underlying topological space.

That shapes may be calculated using hypercovers, as suggested in the introductory discussion, follows from descent and the fact that $\pi_!$ is a left adjoint, and thus commutes with colimits.

E2: The total singular complex functor $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S}) \leftarrow \mathbf{Top}$ factors through the functor $u^* : \mathbf{Diff}^0 \leftarrow \mathbf{Top}$ induced from the inclusion $u : \mathbf{Cart}^0 \hookrightarrow \mathbf{Top}$. Let $U_\bullet \to X$ be a hypercover of topological spaces, then it is not hard to show that $u^* U_\bullet \to u^* X$ is a hypercover of sheaves, so that $u^* X$ is a homotopy colimit of $u^* U_\bullet$ (and thus X, a homotopy colimit of $U_\bullet$), as $\pi_!$ preserves colimits, so that we recover Dugger and Isaksen's hypercover theorem.

Similarly, if $E \to B$ is a topological principal G bundle, then $u^* E \to u^* B$ is a principal $u^* G$ bundle in $\mathbf{Diff}^0$, so that B is a homotopy quotient of E. This fact is often invoked without comment; classical proofs don't seem to be well-known (e.g., [May75, §7 & §8]) and are very technical.

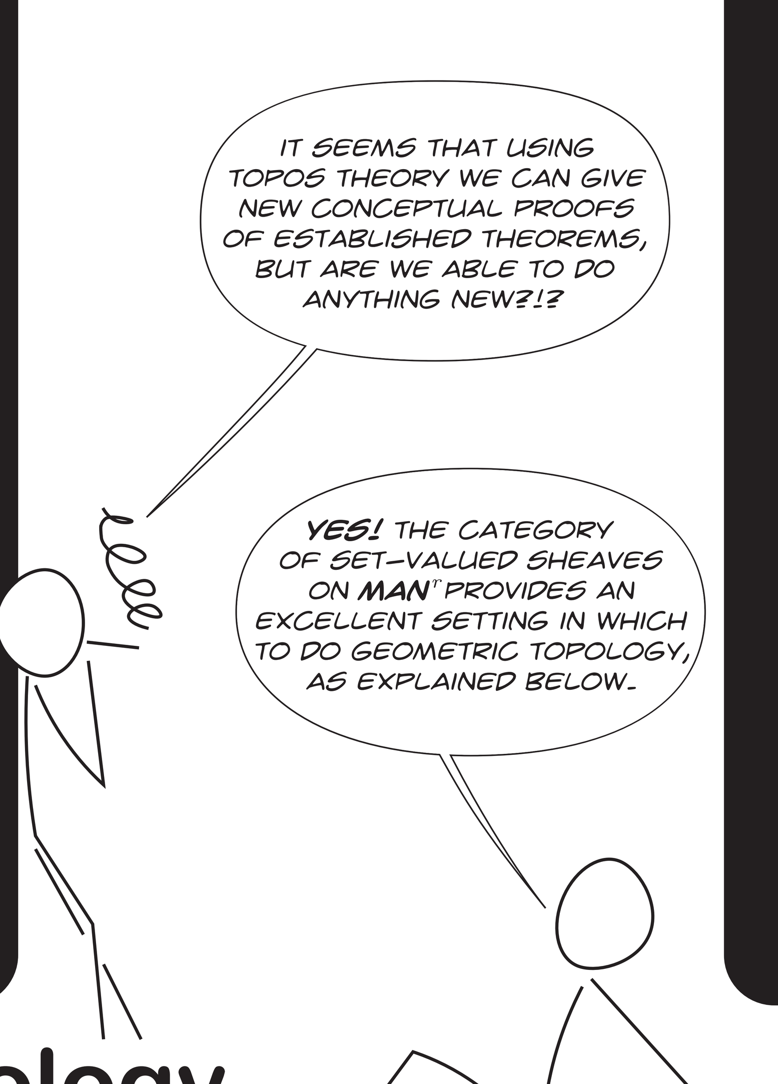
F: Homotopical calculi on differentiable sheaves

Let A be a closed smooth manifold, and X, any smooth manifold without boundary, then $\mathbf{Diff}^\infty(A, X)$ admits the structure of a Fréchet manifold — equivalent to the internal mapping sheaf $\underline{\mathbf{Diff}}^\infty(A, X)$ — and it is a folk theorem that $\underline{\mathbf{Diff}}^\infty(A, X)$ models $\mathcal{S}(\pi_! A, \pi_! X)$. The shape functor $\pi_! : \mathbf{Diff}^r \to \mathcal{S}$ preserves finite products, so that for any sheaves A, X we obtain a comparison map $\pi_! \underline{\mathbf{Diff}}^\infty(A, X) \to \mathcal{S}(\pi_! A, \pi_! X)$; we prove the following generalisation of [BEBP19, Th. 1.1]:

Theorem (C.). **Let A be a nice, possibly infinite dimension, manifold[1], and X any sheaf, then $\pi_! \underline{\mathbf{Diff}}^\infty(A, X) \to \mathcal{S}(\pi_! A, \pi_! X)$ is an isomorphism.**

Call A **formally cofibrant** if $\pi_! \underline{\mathbf{Diff}}^\infty(A, X) \to \mathcal{S}(\pi_! A, \pi_! X)$ is an isomorphism for all X, then the proof idea is as follows: Let $S \to D$ be a map between formally cofibrant objects such that $\underline{\mathbf{Diff}}^\infty(D, X) \to \underline{\mathbf{Diff}}^\infty(S, X)$ is sharp for all X, then for any "attaching" map $f : S \to A$ the pushout $A \cup_f D$ is also formally cofibrant. Moreover, any sheaf which is $\mathbf{R}$-homotopy equivalent to a formally cofibrant sheaf is itself formally cofibrant; in particular, any $\mathbf{R}$-contractible sheaf is formally cofibrant. Kihara endows the simplices Δ^n with a non-standard smooth structure which coincides with the usual smooth structure on $\mathring{\Delta}^n$, but for which the horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ admit retracts. The goal is then to show that the morphism $\underline{\mathbf{Diff}}^\infty(\Delta^n, X) \to \underline{\mathbf{Diff}}^\infty(\partial \Delta^n, X)$ is sharp for all X, and that any nice manifold is $\mathbf{R}$-homotopy equivalent to a simplicial complex built from Kihara's simplices. Given **E1**, one might be tempted to show that the above morphism is sharp by showing it is a fibration in the model structure transferred using Kihara's simplices; unfortunately, this is equivalent to showing that this model structure is Cartesian which is false. Instead, using the cube category one can construct a functor $\square \to \mathrm{Pro}(\mathbf{Diff}^r)$, which induces a fibration structure in which the above morphism is indeed sharp.

[1] See [Kih20, Th. 11.1]; any finite dimensional paracompact Hausdorff manifold is nice.



applications to geometric topology

The restriction of the shape functor to set-valued sheaves $\pi_! \mathbf{Diff}^r_{\leq 0} \to \mathcal{S}$ is still a localisation, so that we obtain a model for homotopy types, which in the setting of geometric topology is in many way superior to the model provided by topological spaces (a fact already exploited in [GTMW09] and [Kup19]). E.g, many important spaces in geometric topology such as mapping spaces or spaces of embedded manifolds naturally come with a smooth structure. When trying to encode these spaces as topological spaces one has to try to find topologies which approximate these smooth structures, which is often very intricate. On the other hand, these spaces are frequently completely straightforward to write down as objects in $\mathbf{Diff}^r_{\leq 0}$. Moreover, $\mathbf{Diff}^r_{\leq 0}$ has excellent formal properties, e.g., the inclusion functor $\mathbf{Diff}^r_{\leq 0} \hookrightarrow \mathbf{Diff}^r$ commutes with filtered colimits, so that filtered colimits are homotopy colimits in $\mathbf{Diff}^r_{\leq 0}$.

We illustrate the above considerations with the calculation of $\pi_! \mathrm{Conf}(\mathbf{R}^n)$ by making precise an idea originally due to Segal ([Seg79, Prop. 3.1]). Here, $\mathrm{Conf}(\mathbf{R}^n)$ is the space of finite unordered configurations in $\mathbf{R}^n$ (without collisions), which in $\mathbf{Diff}^r_{\leq 0}$ may simply be defined as the sheaf which associates to any manifold M the set of submanifolds $C \subseteq M \times \mathbf{R}^n$ such that the map $C \to M$ is a submersion with 0-dimensional fibres (thus, avoiding the difficulty of topologically encoding how points in $\mathbf{R}^n$ may "escape to infinity"). For every $\varepsilon > 0$ denote by $\mathrm{Conf}_\varepsilon(\mathbf{R}^n)$ (resp. $\mathrm{Conf}_{\leq 1}(\mathbf{R}^n)$) the subspace of $\mathrm{Conf}(\mathbf{R}^n)$ consisting of those configurations containing at most one point in $\mathring{B}_\varepsilon(0)$ (resp., all of $\mathbf{R}^n$), then $\mathrm{Conf}_{\leq 1}(\mathbf{R}^n)$ may be exhibited as a retract of $\mathrm{Conf}_\varepsilon(\mathbf{R}^n)$ by pushing all points outside of $\mathring{B}_\varepsilon(0)$ in any configuration in $\mathrm{Conf}_\varepsilon(\mathbf{R}^n)$ off to infinity. Moreover, $\mathrm{Conf}_{\leq 1}(\mathbf{R}^n)$ is $\mathbf{R}$-homotopy equivalent to S^n, as $\mathrm{Conf}_{\leq 1}(\mathbf{R}^n)$ is essentially the one-point-compactification of $\mathbf{R}^n$. Finally we have $\mathrm{colim}_{\varepsilon > 0} \mathrm{Conf}_\varepsilon(\mathbf{R}^n) = \mathrm{Conf}(\mathbf{R}^n)$, so that

$$\pi_! \mathrm{Conf}(\mathbf{R}^n) = \pi_! \operatorname*{colim}_{\varepsilon > 0} \mathrm{Conf}_\varepsilon(\mathbf{R}^n) = \operatorname*{colim}_{\varepsilon > 0} \pi_! \mathrm{Conf}_\varepsilon(\mathbf{R}^n) = \operatorname*{colim}_{\varepsilon > 0} \pi_! S^n = \pi_! S^n.$$

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