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Pro-algebraic Resolutions of Regular Schemes

by

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Master Thesis

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Introduction

In a letter to J-P. Serre dated 9.8.1960 A. Grothendieck writes that he has discovered how to associate to any smooth scheme X over an arbitrary field k a chain complex of commutative pro-algebraic k-groups

$$0 \to J_n \to \cdots \to J_1 \to J_0 \to 0$$

such that for any commutative algebraic k-group G there is a canonical isomorphism $H^i(X, G) \cong H^i(\mathrm{Hom}(J_*, G))$ for all $i \geq 0$ (see [CS01, p. 109]). He only provides a brief sketch of how this is to be achieved. In this master thesis we give a complete construction of this chain complex; we call it the *pro-algebraic resolution of* X. We only assume that X is regular and connected, but we make the extra assumption that k is algebraically closed of characteristic 0.

This master thesis is organised into two parts: Part I contains the preliminaries and Part II the actual construction of the pro-algebraic resolution.

Part I comprises Chapters 1 - 3. Given any category one may form its so-called pro-completion; this is a new category, which in conceptual terms is obtained by formally adding all filtered limits (these do in general not coincide with the filtered limits already present in the given category); this is a useful way of enlarging a category, which we will use in Part II to enlarge the category $\mathbf{AGS}_k$ of commutative algebraic k-groups. We discuss pro-completions in Chapter 1. In the second chapter we discuss group schemes over k, with an emphasis on the following three points: Firstly, we will see that various subcategories of the category of commutative k-groups are Abelian. Secondly, we give a detailed account of the structure theory of commutative algebraic k-groups; this will be important for reducing all the proofs in Part II to certain special cases. Finally, we discuss Abelian sheaves represented by commutative k-groups. In the third chapter we discuss local cohomology; special attention is given to the local cohomology of local rings as well as the so called Cousin-complex of flasque sheaves; these are the technical tools which form the core of Chapter 4 in Part II.

Part II consists of Chapters 4 & 5. For every commutative algebraic k-group G we denote by G_X the Abelian sheaf $X \supseteq U \mapsto \mathbf{Sch}_k(U, G)$; the groups $H^i(X, G)$ $(i \geq 0)$ are the cohomology groups of this sheaf. We would like to find a chain complex J_* of commutative algebraic k-group from which we can recover these cohomology groups, as described above; unfortunately the category $\mathbf{AGS}_k$ is not big enough to contain the constituent groups of such a chain complex, which is why we must consider its pro-completion. The construction of the pro-algebraic resolution of a connected regular scheme consists of two main steps corresponding to Chapters 4 & 5. In Chapter 4 we show that for each commutative algebraic k-group G the sheaf G_X satisfies the

so-called Cohen-Macaulay condition; this allows us to apply the theory developed in §3.2.3 to functorially associate to every commutative algebraic k-group G a flasque resolution $C^*_{X,G}$ of G_X called the Cousin resolution of G_X. In the second step, discussed in Chapter 5, we show that the functor $\mathbf{AGS}_k \to \mathbf{Ab}$ given by $G \mapsto \Gamma(X, C^i_{X,G})$ is pro-representable for every $i \geq 0$.
The first step itself splits into two intermediate steps: In §4.1 we show that checking that G_X is Cohen-Macaulay for every $G \in \mathbf{AGS}_k$ may be reduced to showing that for every $x \in X$ the local cohomology groups $H^i_{\{x\}}(\operatorname{Spec} \mathscr{O}_{X,x}, G)$ vanish for $i \neq \operatorname{ht} x$. The second step, discussed in §4.2, consists in proving that this is indeed the case.
In his letter Grothendieck furthermore makes certain assertions about the structure of the k-groups J_i, notably he claims that they are connected and affine for $i \geq 1$, and unipotent for $i \geq 2$. We prove this in §5.2.
At this point we would like to alert the reader to the following potential source of confusion: Even though the category $\mathbf{AGS}_k$ is Abelian and the functor $\mathbf{AGS}_k \to \mathbf{Ab}$ given by $G \mapsto \mathbf{Sch}_k(X, G)$ is left exact, the functor $G \mapsto H^i_X(X, G)$ is not its derived functor.

Prerequisites

We are assuming competence in basic category theory and homological algebra, as may be found in [Mac98, Ch. I-V & §X.3] and [Sch11b, Ch. 3 & 4] respectively; in particular, we are assuming the reader to be very comfortable with Yoneda's lemma and group objects. We will use some basic sheaf theory on topological spaces in the scope of [MM92, Ch. II] as well as sheaf cohomology on topological spaces; the latter only requires knowledge pertaining to general homological algebra and the fact that one may calculate sheaf cohomology using flasque resolutions. We also require some rudimentary knowledge of Grothendieck topologies, sheaves on sites, and the cohomology of such sheaves; the material found in [Mil80, Ch. I-III] is more than enough. Finally, rather little algebraic geometry is needed: The main prerequisites correspond to the material in [GW10, Ch. 2-5 & §6.11] and a very superficial understanding of the rest of [GW10].

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Conventions and notation

Linguistic conventions

In order to facilitate visual recognisability we use the following contractions:

- We write "iff" instead of "if and only if".
- We write "w.l.o.g." instead of "without loss of generality".
- We write "w.r.t." instead of "with respect to'.

Editorial conventions

- Propositions stated without proof are marked with the symbol $\square$.

Set theory

- We fix a universe $\mathcal{U}$. We call a category $\mathcal{C}$ *small* if its sets of objects and morphisms belong to $\mathcal{U}$ and *essentially small* if it is equivalent to a small category. From now on we use the term *category* to mean a category whose hom-sets all belong to $\mathcal{U}$ and we use the term *large category* to speak of categories with arbitrary hom-sets.
- We will generally assume that any set encountered in this thesis except for sets of objects of categories belongs to $\mathcal{U}$. We will only need this level of set theoretical precision in Chapter 1 and in the proof of Theorem 5.1.1.

Algebra

- All rings are unitary and commutative (and belong to $\mathcal{U}$).
- The comultiplication homomorphism of any Hopf algebra is always denoted by Δ.

Category theory

- Let $\mathcal{C}$ be a (possibly large) category and let $X, Y \in \mathcal{C}$ be two objects, then the set of morphisms from X to Y is denoted by $\mathcal{C}(X, Y)$.
- We use the following notation for various categories (some of these will be defined formally in the text).

- **Set** denotes the category of sets in $\mathcal{U}$.
- $\mathbf{Set}^{\text{fin}}$ denotes the category of finite sets in $\mathcal{U}$.
- **Cat** denotes the large 2-category of categories.
- **Grp** denotes the category of groups in $\mathcal{U}$.
- $\mathbf{Grp}^{\text{fp}}$ denotes the category of finitely presented groups in $\mathcal{U}$.
- **Ab** denotes the category of Abelian groups in $\mathcal{U}$.
- **Top** denotes the category of topological spaces in $\mathcal{U}$.
- For any topological space X in $\mathcal{U}$ we denote by $\mathbf{Ouv}_X$ the category of open sets of X.
- Let k be a ring, then $\mathbf{Alg}_k$ denotes the category of algebras over k in $\mathcal{U}$.
- Let k be a ring, then $\mathbf{Alg}_k^{\text{fp}}$ denotes the category of finitely presented algebras over k in $\mathcal{U}$.
- Let $(X, \mathscr{O}_X)$ be a ringed space in $\mathcal{U}$, then $\mathbf{Mod}_{\mathscr{O}_X}$ denotes the category of $\mathscr{O}_X$-modules in $\mathcal{U}$.
- Let k be a ring, then $\mathbf{Aff}k$ denotes the category of affine k-schemes in $\mathcal{U}$.
- Let S be a scheme in $\mathcal{U}$, then $\mathbf{Sch}_S$ denotes the category of S-schemes in $\mathcal{U}$.
- Let k be a field, then $\mathbf{AGS}_k$ denotes the category of algebraic k-groups in $\mathcal{U}$.
- Let k be a field, then $\mathbf{AAGS}_k$ denotes the category of affine algebraic k-groups in $\mathcal{U}$.
- Let k be a field, then $\mathbf{PAGS}_k$ denotes the pro-completion of the category of algebraic k-groups in $\mathcal{U}$.
- Let k be a field, then $\mathbf{PAAGS}_k$ denotes the category of affine k-groups in $\mathcal{U}$.

- Let $\mathcal{C}$ be a category, then a contravariant functor $\mathcal{C}^{\text{op}} \to \mathbf{Set}$ is called a *presheaf on* $\mathcal{C}$ and the category of presheaves on $\mathcal{C}$ is denoted by $\widehat{\mathcal{C}}$. Dually, a covariant functor $\mathcal{C} \to \mathbf{Set}$ is called a *copresheaf* and the opposite category of the category of coprosheaves on $\mathcal{C}$ is denoted by $\check{\mathcal{C}}$.

- The functor $よ_{\mathcal{C}} : \mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$ (or simply $よ$ when $\mathcal{C}$ is clear from context) denotes the Yoneda embedding. The functor $\text{こ}_{\mathcal{C}} : \mathcal{C} \hookrightarrow \check{\mathcal{C}}$ (or simply こ when $\mathcal{C}$ is clear from context) denotes the co-Yoneda embedding. We will often view $\mathcal{C}$ as a subcategory of $\widehat{\mathcal{C}}$ and $\check{\mathcal{C}}$.

- A functor is called *left (right) exact* if it commutes with all finite limits (colimits). Warning: This definition is less general than the one used in [KS06] (which we cite heavily); the two definitions agree if the domain category admits finite limits (colimits).

- Let $\mathcal{C}$ be a category, then $\text{Grp}(\mathcal{C})$ denotes the category of group objects in $\mathcal{C}$ and $\text{CoGrp}(\mathcal{C})$ denotes the category of cogroup objects in $\mathcal{C}$.

- If a category admits an initial object, this object will denoted by $\varnothing$, and if the category admits a final object, this object will be denoted by 1.

Sheaf theory

- By an *Abelian sheaf* we mean a sheaf of Abelian groups.
- Let X be a topological space, then the category of presheaves and the category of sheaves on X are denoted by $\mathbf{PSh}_X$ and $\mathbf{Sh}_X$ respectively; we will use this notation both for (pre-)sheaves of sets and for Abelian (pre-)sheaves, with the correct interpretation being clear from context. Furthermore $\mathbf{PSh}^s_X$ and $\mathbf{PSh}^f_X$ denote the categories of separated and flasque presheaves on X respectively.
- Let $(\mathcal{C}, J)$ be a site, then the category of sheaves on $(\mathcal{C}, J)$ is denoted by $\widetilde{\mathcal{C}}_J$.

Algebraic geometry

- Let k be a ring in $\mathcal{U}$, then we denote by $\mathscr{O} : \mathbf{Sch}_k^{\mathrm{op}} \to \mathbf{Aff}k$ the functor which assigns to any k-scheme the k-algebra $\Gamma(X, \mathscr{O}_X)$.

Part I

Preliminaries

Chapter 1

Pro-completions

Given a category $\mathcal{C}$, then, conceptually speaking, its pro-completion is the category obtained by formally adjoining all filtered limits to $\mathcal{C}$ and its ind-completion is the category obtained by formally adjoining all filtered colimits. These "adjoined" filtered (co-)limits rarely coincide with any filtered (co-)limits already present in $\mathcal{C}$ (see [KS06, §6.2]).

There are (at least) two reasons why it may be useful to consider pro- and ind-completions. Sometimes a category does not have "enough" objects and its pro- or ind-completion is a good way of enlarging it; sometimes this enlarged version itself admits a concrete description (see Example 1.2.4). In other cases it may turn out that a category under consideration is the pro- or ind-completion of a category satisfying certain properties, from which one may in turn infer certain properties of the given category, e.g. the ind-completion of an essentially small category admitting finite colimits is both well-powered and co-well-powered and admits both small limits and colimits (see [KS06, Prop. 6.1.18.iii] and [AR94, Rm. 1.56]); these are strong properties, e.g. a functor from such a category to any other category has a right adjoint iff it commutes with small colimits (see [Mac98, p. 130]).

This chapter is one of the only places in this thesis where it is important that we distinguish between small and large categories; the reader may want to quickly review our set theoretical conventions on Page v.

Finally we note that the title of this chapter is a bit misleading as we will mainly consider ind-completions (except in the very last subsection), as these are somewhat less fiddly to work with.

The standard reference for the material in this chapter is [KS06, Ch. 6].

1.1 Basic definitions and properties

1.1.1 Categories of presheaves

In order to understand ind- and pro-completions we must first better understand (large) categories of presheaves and copresheaves. These categories will also serve as prototypes of ind- and pro-completions. Throughout this section $\mathcal{C}$ denotes a category.

We will only work with $\widehat{\mathcal{C}}$, as all analogous statements pertaining to $\check{\mathcal{C}}$ may be obtained by

considering the appropriate opposite categories.
We wish to show now that if $\mathcal{C}$ is small, then $\widehat{\mathcal{C}}$ may be thought of as the category obtained by formally adjoining all colimits to $\mathcal{C}$ and $\check{\mathcal{C}}$ as the category obtained by formally adjoining all limits to $\mathcal{C}$.

Notation 1.1.1. Let $\mathcal{I}$ be a small category and let $A : \mathcal{I} \to \mathcal{C}$ be a functor, then we write "$\varinjlim$"A for the colimit of $よ \circ A$, and similarly we write "$\varprojlim$"A for the limit of $\rotatebox{90}{よ} \circ A$. ⌟

Let F be a presheaf on $\mathcal{C}$ and denote by $\mathcal{C}_F$ the slice category of $\mathcal{C}$ over F, that is, the category whose objects are pairs (X, η), where X is an object in $\mathcal{C}$ and η is a natural transformation $\mathcal{C}(\_, X) \to F$, and whose morphisms $(X_1, \eta_1) \to (X_2, \eta_2)$ are morphisms $X_1 \to X_2$ in $\mathcal{C}$ making the resulting triangle

$$\begin{array}{ccc} \mathcal{C}(\_, X_1) & \Longrightarrow & \mathcal{C}(\_, X_2) \\ & {\scriptstyle \eta_1}\searrow \quad \swarrow{\scriptstyle \eta_2} & \\ & F & \end{array}$$

commute[1]. The functor of F then forms the vertex of the cocone

$$\{\eta : \mathcal{C}(\_, X) \Rightarrow F\}_{(X,\eta)\in\mathcal{C}_F} . \tag{1.1}$$

Proposition 1.1.2. [Mac98, Th. III.7.1] *For any presheaf F on $\mathcal{C}$ the cocone (1.1) is univesal, i.e. there is a canonical isomorphism*

$$F \cong \text{"}\varinjlim\text{"}(\mathcal{C}_F \to \mathcal{C}).$$

□

Remark 1.1.3. The (large) category $\widehat{\mathcal{C}}$ does not contain all colimits, only all *small* colimits. It is thus not true a priori that the colimit of $\mathcal{C}_F \to \widehat{\mathcal{C}}$ exists. ⌟

Proposition 1.1.4. *Assume that $\mathcal{C}$ is small and let $\mathcal{A}$ be a category containing all small colimits, then for any functor $F : \mathcal{C} \to \mathcal{A}$ there exists a pair consisting of a functor $\widehat{F} : \widehat{\mathcal{C}} \to \mathcal{A}$ commuting with all small colimits together with a natural isomorphism $F \Rightarrow \widehat{F} \circ よ$; this pair is unique up to natural isomorphism, that is, for any other pair consisting of a functor $\widehat{G} : \widehat{\mathcal{C}} \to \mathcal{A}$ commuting with all small colimits together with a natural isomorphism $F \Rightarrow \widehat{G} \circ よ$, there exists a unique natural isomorphism $\widehat{F} \Rightarrow \widehat{G}$ such that the resulting triangle*

$$\begin{array}{ccc} \widehat{F} \circ よ & \Longrightarrow & \widehat{G} \circ よ \\ & \nwarrow \quad \nearrow & \\ & F & \end{array}$$

commutes.

[1] Incidentally, the category $\mathcal{C}_F$ is canonically isomorphic to the category of elements of F.

Idea of proof. The pair $(\widehat{F}, \eta)$ is simply the left Kan extension of $F : \mathcal{C} \to \mathcal{A}$ along $よ : \mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$, and one has to show that $\widehat{F}$ commutes with colimits and that η is an isomorphism. For details see [KS06, Prop. 2.7.1]. $\square$

We finish by stating an important property of representable functors in categories of presheaves.

Proposition 1.1.5. *For any object $X \in \mathcal{C}$ the functor $\widehat{\mathcal{C}}(よ(X), \_) : \widehat{\mathcal{C}} \to$ **Set** commutes with small colimits.*

Proof. Let $\mathcal{I}$ be a small category and let $A : \mathcal{I} \to \widehat{\mathcal{C}}$ be a functor, then we have the following canonical isomorphisms

$$\widehat{\mathcal{C}}(よ(X), \varinjlim_{i \in \mathcal{I}} A_i) \cong (\varinjlim_{i \in \mathcal{I}} A_i)(X) \cong \varinjlim_{i \in \mathcal{I}} A_i(X) \cong \varinjlim_{i \in \mathcal{I}} \widehat{\mathcal{C}}(よ(X), A_i),$$

where the first and the last isomorphisms are due to Yoneda's lemma, and the second to the fact that colimits in categories of functors may be computed objectwise. $\square$

1.1.2 Pro-completions

Definition 1.1.6.

1. An *ind-object in* $\mathcal{C}$ or an *ind-representable object in* $\widehat{\mathcal{C}}$ is an object in $\widehat{\mathcal{C}}$ which is isomorphic to "$\varinjlim$"A for some functor A from a small filtered category to $\mathcal{C}$.
2. We denote by $\mathrm{Ind}(\mathcal{C})$ the full subcategory of $\widehat{\mathcal{C}}$ spanned by ind-objects in $\mathcal{C}$ and call it the *ind-completion of* $\mathcal{C}$.
3. A *pro-object in* $\mathcal{C}$ or a *pro-representable object in* $\check{\mathcal{C}}$ is an object in $\check{\mathcal{C}}$ which is isomorphic to "$\varprojlim$"A for some functor A from a small cofiltered category to $\mathcal{C}$.
4. We denote by $\mathrm{Pro}(\mathcal{C})$ the full subcategory of $\check{\mathcal{C}}$ spanned by pro-objects in $\mathcal{C}$ and call it the *pro-completion of* $\mathcal{C}$.

⌟

We see that the ind-completion of a category is similar to its (large) category of presheaves. We begin by citing an immediate advantage of its ind-completions over its (large) category of presheaves.

Proposition 1.1.7. $\mathrm{Ind}(\mathcal{C})$ *is a category, i.e. its hom-sets belong to* $\mathcal{U}$. ⌟

Proof. Let $\mathcal{I}$ and $\mathcal{J}$ be filtered categories, let $A : \mathcal{I} \to \mathcal{C}$ and $B : \mathcal{J} \to \mathcal{C}$ be two functors, then we have

$$\begin{aligned} \mathrm{Ind}(\mathcal{C})(\text{“}\varinjlim\text{”}A, \text{“}\varinjlim\text{”}B) &= \widehat{\mathcal{C}}(\text{“}\varinjlim\text{”}A, \text{“}\varinjlim\text{”}B) \\ &\cong \varprojlim_{i \in \mathcal{I}} \widehat{\mathcal{C}}(よ(A(i)), \text{“}\varinjlim\text{”}B) \\ &\cong \varprojlim_{i \in \mathcal{I}} \varinjlim_{j \in \mathcal{J}} \mathcal{C}(よ(A(i)), よ(B(j))), \end{aligned}$$

where the penultimate isomorphism is due to the universal property of colimits and the last isomorphism follows from Proposition 1.1.5. □

Proposition 1.1.8. [KS06, Th. 6.1.8] *The category* $\mathrm{Ind}(\mathcal{C})$ *admits small filtered colimits and the canonical functor* $\mathrm{Ind}(\mathcal{C}) \hookrightarrow \widehat{\mathcal{C}}$ *commutes with such colimits.* □

The ind-completion of a category satisfies an analogous universal property as the category of presheaves of a small category.

Proposition 1.1.9. *Let* $\mathcal{A}$ *be a category containing all small filtered colimits, then for any functor* $F : \mathcal{C} \to \mathcal{A}$ *there exists a pair consisting of a functor* $F' : \mathrm{Ind}(\mathcal{C}) \to \mathcal{A}$ *commuting with all small filtered colimits together with a natural isomorphism* $F \Rightarrow F' \circ \text{よ}$*; this pair is unique up to natural isomorphism, that is, for any other pair consisting of a functor* $G' : \mathrm{Ind}(\mathcal{C}) \to \mathcal{A}$ *commuting with all small filtered colimits together with a natural isomorphism* $F \Rightarrow G' \circ \text{よ}$ *there exists a unique natural isomorphism* $F' \Rightarrow G'$ *such that the resulting triangle*

$$\begin{array}{ccc} F' \circ \text{よ} & \Longrightarrow & G' \circ \text{よ} \\ & \nwarrow\!\!\!\!\nwarrow \quad \nearrow\!\!\!\!\nearrow & \\ & F & \end{array}$$

commutes.

Idea of proof. The idea of the proof is the same as for the proof of Proposition 1.1.4. For details see [KS06, Cor. 6.3.2]. □

We now give a different characterisation of the presheaves on $\mathcal{C}$ which belong to $\mathrm{Ind}(\mathcal{C})$, provided that $\mathcal{C}$ contains finite colimits. Any representable presheaf on $\mathcal{C}$ obviously takes small colimits in $\mathcal{C}$ to small limits; this is no longer true for a presheaf in $\mathrm{Ind}(\mathcal{C})$, but such a presheaf is still left exact, that is, it takes finite colimits to finite limits. This is easily verified: Let $\mathcal{I}$ be a small filtered category and $\mathcal{K}$ a finite category, and consider two functors $A : \mathcal{I} \to \mathcal{C}$ and $F : \mathcal{K} \to \mathcal{C}$, then we have the following sequence of canonical isomorphisms:

$$\begin{aligned} \mathrm{Ind}(\mathcal{C})(\varinjlim F, \text{“}\varinjlim\text{”} A) &\cong \varinjlim_{i \in \mathcal{I}} \mathcal{C}(\varinjlim F, A(i)) \\ &\cong \varinjlim_{i \in \mathcal{I}} \varprojlim_{k \in \mathcal{K}} \mathcal{C}(F(k), A(i)) \\ &\cong \varprojlim_{k \in \mathcal{K}} \varinjlim_{i \in \mathcal{I}} \mathcal{C}(F(k), A(i)) \\ &\cong \varprojlim_{k \in \mathcal{K}} \mathrm{Ind}(\mathcal{C})(F(k), \text{“}\varinjlim\text{”} A), \end{aligned}$$

where the first and the last isomorphisms follow from Proposition 1.1.5, the second isomorphism is due to the universal property of colimits, and the third is due to the fact that filtered colimits commute with filtered limits in **Set** (see [KS06, Th. 3.1.6]). It turns out that the converse is almost true.

Proposition 1.1.10. [KS06, Prop. 6.1.7] *Assume that $\mathcal{C}$ admits finite colimits, then a presheaf F on $\mathcal{C}$ is ind-representable iff it is left exact and $\mathcal{C}_F$ is cofinally small.* □

Corollary 1.1.11. *If $\mathcal{C}$ is essentially small and admits small colimits, then a presheaf F on $\mathcal{C}$ is Ind-representable iff is left exact.* □

Remark 1.1.12. There is a well developed theory of categories which satisfy smallness conditions similar to that of being the ind-completions of an essentially small category. See [AR94] for a very readable introduction. ⌟

1.1.3 The relationship between (co-)limits in $\mathcal{C}$ and $\mathrm{Pro}(\mathcal{C})$

We have already seen in Proposition 1.1.8 that $\mathrm{Ind}(\mathcal{C})$ admits all filtered colimits. It turns out that the other types of limits and colimits which are admitted by $\mathrm{Ind}(\mathcal{C})$ depend on the types of limits and colimits admitted by $\mathcal{C}$. We summarise these dependencies which are scattered throughout [KS06, §6.1] in the following table:

If $\mathcal{C}$ admits	then $\mathrm{Ind}(\mathcal{C})$ admits
finite limits	finite limits.
small limits	small limits.
coequalisers	coequalisers.
finite coproducts	small coproducts.
finite colimits	small colimits.

(1.2)

Furthermore we have the following proposition:

Proposition 1.1.13. *The canonical embedding $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ is exact.*

Proof. The canonical embedding $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ commutes with small limits because $\text{よ} : \mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$ does. For right exactness see [KS06, Cor. 6.1.6]. □

1.1.4 Characterising pro-completions

We saw in Proposition 1.1.5 that for any presheaf F on the category $\mathcal{C}$ the corresponding presheaf $\widehat{\mathcal{C}}(F, \_)$ on $\widehat{\mathcal{C}}$ commutes with small colimits if F is representable. We now discuss a class of objects in any category admitting finite limits which satisfy a similar property.

Definition 1.1.14. Assume that $\mathcal{C}$ admits small filtered limits, then an object $X \in \mathcal{C}$ is called *compact* if for any filtered category $\mathcal{I}$ and any functor $A : \mathcal{I} \to \mathcal{C}$ the canonical map

$$\varinjlim \mathcal{C}(X, A) \to \mathcal{C}(X, \varinjlim A)$$

is an isomorphism. ⌟

As the term "compact" suggests, compact objects are "small" in an appropriate sense; it is easily checked that a necessary condition for $X \in \mathcal{C}$ to be compact is that for any small filtered category $\mathcal{I}$ and any functor $A : \mathcal{I} \to \mathcal{C}$ every morphism $X \to \varinjlim A$ factors through a morphism

$A(i) \to \varinjlim A$ for some $i \in \mathcal{I}$.
We only give one example of compact objects right now, deferring further examples to §1.2.

Example 1.1.15. Let X be a topological space, then an open set in $\mathbf{Ouv}_X$ is compact in the sense of Definition 1.1.14 iff it is compact in the topological sense. This is not hard to check, so we leave it as an exercise. ⌟

Recall that $\mathrm{Ind}(\mathcal{C})$ admits all small filtered colimits (see Proposition 1.1.8). We see by Proposition 1.1.5 that the objects in $\mathcal{C}$ are compact in $\mathrm{Ind}(\mathcal{C})$ and so that by definition every object in $\mathrm{Ind}(\mathcal{C})$ is a filtered colimit of compact objects. As a corollary of the following proposition we obtain a converse in a certain sense.

Proposition 1.1.16. [KS06, Prop. 6.3.4] *Let $\mathcal{P}$ be a category and $F : \mathcal{P} \to \mathcal{C}$ a functor. If*

(a) $\mathcal{C}$ admits small filtered colimits,

(b) F is fully faithful, and

(c) any object in the (essential) image of F is compact,

then the functor $\mathrm{Ind}(\mathcal{P}) \to \mathcal{C}$ *obtained by the universal property of* $\mathcal{P} \to \mathrm{Ind}(\mathcal{P})$ *(see Proposition 1.1.9) is fully faithful.* □

Corollary 1.1.17. *Denote by $\mathcal{C}_c$ the full subcategory of $\mathcal{C}$ spanned by compact objects. If $\mathcal{C}$ admits all small filtered colimits and if every object in $\mathcal{C}$ is the small filtered colimit of compact objects then the canonical functor*

$$\mathrm{Ind}(\mathcal{C}_c) \to \mathcal{C}$$

is an equivalence of categories. □

Proof. By the previous proposition the functor is fully faithful. As the functor commutes with filtered colimits, it is also essentially surjective by assumption. □

Remark 1.1.18. While all objects in $\mathcal{C}$ are compact in $\mathrm{Ind}(\mathcal{C})$ the converse is true iff $\mathcal{C}$ is idempotent complete (see [KS06, Exercise 6.1]). Many examples of categories are idempotent complete, as a sufficient condition is to admit equalisers or coequalisers; in particular, all examples considered in this chapter are idempotent complete. ⌟

1.2 Examples

In this section we study examples of pro- and ind-completions of various categories. All characterisations of pro- and ind-completions in the following examples may be obtained by applying Corollary 1.1.17.

Example 1.2.1. The compact objects in **Set** are exactly the finite sets, so that **Set** is the ind-completion of $\mathbf{Set}^{\mathrm{fin}}$. This is easily verified and left as an exercise. ⌟

Example 1.2.2. Let k-be a ring, then the compact objects in $\mathbf{Alg}_k$ are exactly the finitely presented k-algebras, so that $\mathbf{Alg}_k$ is the ind-completion of $\mathbf{Alg}_k^{\mathrm{fp}}$. This is also easily verified and left as an exercise. ⌟

Remark 1.2.3. The result in the previous two examples holds in much greater generality: In any variety of algebras, the compact objects are exactly the finitely presented algebras (see [AR94, Cor. 1.3]). Thus this result is true for example for the category of groups or the category of modules over a ring. ⌟

Example 1.2.4. The pro-completion of $\mathbf{Set}^{\mathrm{fin}}$ is equivalent to the category of totally disconnected compact Hausdorff spaces; such spaces are called *profinite topological spaces.* ⌟

1.3 Algebraic structures in pro-completions

As we will be discussing pro-algebraic k-groups, we need to understand how group structures in a category and pro-completions interact. We will, as usual, consider the somewhat easier dual structures, that is, ind-completions and cogroup objects.

Lemma 1.3.1. *Assume that $\mathcal{C}$ admits finite coproducts. Let $\mathcal{I}$ be a small category, let $F : \mathcal{I} \to \mathrm{CoGrp}(\mathcal{C})$ be a functor, and assume that the colimit of the composition of F with the forgetful functor $\mathrm{CoGrp}(\mathcal{C}) \to \mathcal{C}$ exists, then the colimit of F exists and is given by $\varinjlim F$ with the cocomposition, counit and coinverse morphisms induced by the universal property of colimits.*

Sketch of proof. All the properties characterising $\varinjlim F$ together with the induced morphisms $\varinjlim F \to \varinjlim F \sqcup \varinjlim F$ and $\varinjlim F \to \varnothing$ as a cogroup object may be checked using the universal property of $\varinjlim F$ (in $\mathcal{C}$). We will show the commutativity of the diagram expressing that the counit morphism indeed acts as a counit, which is the least trivial part of the proof, and leave the rest to the reader. For every $i \in \mathcal{I}$ consider the diagram

$$\begin{array}{ccccccc}
\varinjlim F & \longrightarrow & \varinjlim F \sqcup \varinjlim F & \longrightarrow & \varnothing \sqcup \varinjlim F & \longleftarrow & \varinjlim F \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
F(i) & \longrightarrow & F(i) \sqcup F(i) & \longrightarrow & \varnothing \sqcup F(i) & \longrightarrow & F(i),
\end{array}$$

then we must show that composition of the leftmost two arrows in the top row is equal to the canonical morphism $\varinjlim F \to \varnothing \sqcup \varinjlim F$. This is equivalent to showing that the composition of the leftmost two arrows in the top row with the canonical morphism $F(i) \to \varinjlim F$ is equal to the morphism obtained by composing $F(i) \to \varinjlim F$ and $\varinjlim F(i) \to \varnothing \sqcup \varinjlim F$, but this follows from the commutativity of the three squares in the above diagram and the fact that the composition of the bottom three arrows is equal to $\mathrm{id} : F(i) \to F(i)$. □

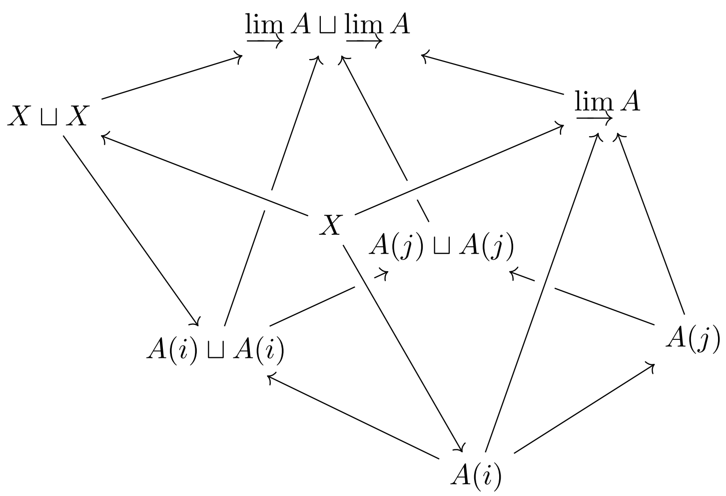
Corollary 1.3.2. *If $\mathcal{C}$ admits colimits indexed by a category $\mathcal{I}$, then so does* $\mathrm{CoGrp}(\mathcal{C})$. □

As the functor $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ is right exact (see Proposition 1.1.13) it induces a functor $\mathrm{CoGrp}(\mathcal{C}) \to \mathrm{CoGrp}(\mathrm{Ind}(\mathcal{C}))$. By Corollary 1.3.2 the category $\mathrm{CoGrp}(\mathrm{Ind}(\mathcal{C}))$ admits filtered colimits so by the universal property of $\mathrm{CoGrp}(\mathcal{C}) \to \mathrm{Ind}(\mathrm{CoGrp}(\mathcal{C}))$ (see Proposition 1.1.9) we obtain the canonical functor

$$\mathrm{Ind}(\mathrm{CoGrp}(\mathcal{C})) \to \mathrm{CoGrp}(\mathrm{Ind}(\mathcal{C})). \tag{1.3}$$

Proposition 1.3.3. *The functor (1.3) is fully faithful.*

Proof. As $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ is fully faithful, so is $\mathrm{CoGrp}(\mathcal{C}) \to \mathrm{CoGrp}(\mathrm{Ind}(\mathcal{C}))$; therefore, by Corollary 1.3.2 and Propositions 1.1.16 we only need to show that the objects of $\mathrm{CoGrp}(\mathcal{C})$ are compact in $\mathrm{CoGrp}(\mathrm{Ind}(\mathcal{C}))$. Let $\mathcal{I}$ be a small filtered category and consider a functor $A : \mathcal{I} \to \mathrm{CoGrp}(\mathrm{Ind}(\mathcal{C}))$, then for any object $X \in \mathrm{CoGrp}(\mathcal{C})$ we must show that the canonical map $\varinjlim \mathrm{CoGrp}(\mathrm{Ind}(\mathcal{C}))(X, A) \to \mathrm{CoGrp}(\mathrm{Ind}(\mathcal{C}))(X, \text{“}\varinjlim\text{”} A)$ is a bijection. Injectivity follows directly from the compactness of X in $\mathrm{Ind}(\mathcal{C})$; surjectivity requires a little bit more care. Again by the compactness of X in $\mathrm{Ind}(\mathcal{C})$ every morphism $X \to \varinjlim A$ factors through $A(i) \to \varinjlim A$ for some object $i \in \mathcal{I}$; the resulting morphism $X \to A(i)$ is however not a priori a morphism of cogroup objects in $\mathrm{Ind}(\mathcal{C})$. The morphism $A(i) \sqcup A(i) \to \varinjlim A \sqcup \varinjlim A$ equalises the two morphisms from X to $A(i) \sqcup A(i)$ in (1.4), so by the compactness of X there exists a morphism $i \to j$ in $\mathcal{I}$ such that $A(i) \sqcup A(i) \to A(j) \sqcup A(j)$ equalises these two morphisms. It is now straightforward to check using (1.4) that the morphism $X \to A(j)$ is a morphism of cogroup objects.



(1.4)

□

Remark 1.3.4. Note that Proposition 1.3.3 tells us that every object in $\mathrm{Ind}(\mathrm{CoGrp}(\mathcal{C}))$ may be viewed as a cogroup object, namely in $\mathrm{Ind}(\mathcal{C})$. We are not aware whether every object in $\mathrm{CoGrp}(\mathrm{Ind}(\mathcal{C}))$ may be viewed as the formal filtered colimit of cogroup objects in $\mathcal{C}$, that is whether the functor (1.3) is an equivalence categories. Any $X \in \mathrm{CoGrp}(\mathrm{Ind}(\mathcal{C}))$ is by construction the filtered colimits of objects in $\mathcal{C}$; one would have to show that there exists a filtered category $\mathcal{I}$ and functor $A : \mathcal{I} \to \mathcal{C}$ such that $X \cong \text{“}\varinjlim\text{”} A$, and moreover A factors through the forgetful functor $\mathrm{CoGrp}(\mathcal{C}) \to \mathcal{C}$. We have been able to show this to be the case in all but one of the examples presented in the following subsection, but always for very specific reasons: In

Examples 1.3.5 - 1.3.8 the proof that every cogroup object in $\mathrm{Ind}(\mathcal{C})$ ($\mathrm{Pro}(\mathcal{C})$) is isomorphic to the filtered colimit (limit) of compact cogroup objects relies heavily on the concrete structure of cogroup objects in $\mathrm{Ind}(\mathcal{C})$ ($\mathrm{Pro}(\mathcal{C})$); in Example 1.3.10 one uses the fact that in this case the forgetful functor $\mathrm{CoGrp}(\mathcal{C}) \to \mathcal{C}$ is an isomorphism. Thus these examples do not seem to indicate that (1.3) ought to be an equivalence a priori. ⌟

1.3.1 Examples

Example 1.3.5. The only object in **Set** with the structure of a cogroup object is the empty set, because the only set with a map to the empty set is the empty set itself, so only the empty set may have a counit morphism. The comultiplication and coinverse maps are uniquely determined and are readily seen to satisfy the axioms of a cogroup object. Setting $\mathcal{C} = \mathbf{Set}^{\mathrm{fin}}$ we see that (1.3) is an isomorphism of categories as it is the unique functor from the category containing one object and one morphism to itself. ⌟

Example 1.3.6. [Kan58, §3] The cogroup objects in **Grp** are free groups, and any choice of free basis for a free group uniquely determines the structure of a cogroup object. Let X be a set, $F(X)$ the corresponding free group, and $j_1, j_2 : F(X) \to F(X) * F(X) \cong F(X \sqcup X)$ the canonical injections, then the comultiplication homomorphism on $F(X)$ is determined by sending any $x \in X$ to $j_1(x) \cdot j_2(x)$, the coinverse homomorphism is determined by sending any $x \in X$ to x^{-1}, the counit homomorphism is the unique homomorphism $F(X) \to 1$. By Remark 1.2.3 the compact objects in **Grp** are exactly the finitely presented groups, and it is easily seen that $F(X)$ is finitely presented iff X is finite. The cogroup object $F(X)$ is the colimit of all sub-cogroup-objects of the form $F(X')$, where X' is a finite subset of X, so we see that (1.3) is an equivalence of categories if we set $\mathcal{C} = \mathbf{Grp}^{\mathrm{fp}}$. ⌟

Example 1.3.7. [Mil12, Th. 8.2] Let k be a field and let A be a Hopf k-algebra, then, using representation theory of Hopf k-algebras, one may show that every finite subset of A is contained in a finitely generated (as a k-algebra) Hopf k-subalgebra of A. Thus A is the union of all its finitely generated (as k-algebras) Hopf k-subalgebras, and it is straightforward to check that A is the colimit over the resulting diagram of Hopf k-subalgebras. As k is Noetherian, the notions of finitely presented and finitely generated k-algebras coincide, and we see that if we set $\mathcal{C} = \mathbf{Alg}_k$, then (1.3) is an equivalence of categories. ⌟

Example 1.3.8. A profinite group is a topological group whose underlying topological space is profinite (i.e. totally disconnected and compact; see Example 1.1.17). By [Sha72, Th. 1.1.2] every profinite group is the filtered limit of its finite quotients, so we see that (1.3) is an equivalence of categories for $\mathcal{C} = \mathbf{Set}^{\mathrm{fin}}$.

Profinite groups have important applications. Let k be a field and let $k \hookrightarrow K$ be a Galois extension; if $k \hookrightarrow K$ is infinite, then $\mathrm{Aut}_k(K)$ is no longer the correct object with which to do Galois theory, as not all subgroups of $\mathrm{Aut}_k(K)$ correspond to subextensions of $k \hookrightarrow K$; instead one considers the profinite group obtained as the filtered colimits of all Galois groups of all finite subextensions of $k \hookrightarrow K$. Viewing this pro-object as a topological group one may verify that its underlying group is in fact $\mathrm{Aut}_k(K)$, but only closed subgroups correspond to subextensions. For detailed expositions see [BJ01, §3] and [Sza09, §1]. ⌟

Example 1.3.9. [Fau08, Cor. A.3] Similarly to the previous example, the pro-completion of the category of compact Lie groups is equivalent to the category of compact Hausdorff groups. We are unaware whether (1.3) is an equivalence of categories if we take $\mathcal{C}$ to be the category of compact, r times differentiable manifolds, where $r \in \mathbb{N} \cup \{\infty\} \cup \{\omega\}$. ⌟

Example 1.3.10. The functor (1.3) is an equivalence of categories if $\mathcal{C}$ is an additive category[2]. This follows from the fact that in this case the forgetful functor $\mathrm{CoGrp}(\mathcal{C}) \to \mathcal{C}$ is an isomorphism of categories. ⌟

1.4 Pro-completions of Abelian categories

Throughout this section $\mathcal{A}$ denotes an Abelian category. We will show that $\mathrm{Ind}(\mathcal{A})$ is again Abelian. This is done in two steps: First we show that $\mathrm{Ind}(\mathcal{A})$ is a subcategory of the (large) Abelian category of Abelian additive presheaves on $\mathcal{A}$; by Table (1.2) we see that $\mathrm{Ind}(\mathcal{A})$ admits finite limits and small colimits, so that $\mathrm{Ind}(\mathcal{A})$ is additive; we then only have to show that $\mathrm{Ind}(\mathcal{A})$ satisfies AB 2) (see [Gro57, §1.3]). We begin with three lemmas.

Lemma 1.4.1. [KS06, Prop. 8.2.15] *A functor between additive categories is additive iff it commutes with finite products (or equivalently with finite coproducts).* □

Lemma 1.4.2. *Every presheaf on $\mathcal{A}$ which commutes with finite coproducts factors uniquely through the forgetful functor $U : \mathbf{Ab} \to \mathbf{Set}$.*

Proof. Let F be a presheaf on $\mathcal{A}$ which commutes with finite coproducts, then any object $X \in \mathcal{A}$ has the unique structure of a co-Abelian cogroup with comultiplication

$$X \xrightarrow{\begin{bmatrix}\mathrm{id}_X \\ \mathrm{id}_X\end{bmatrix}} X \oplus X, \tag{1.5}$$

which induces the structure of an Abelian group on $F(X)$, so we see that F factors through $U : \mathbf{Ab} \to \mathbf{Set}$. To see uniqueness we must check that for any other additive functor $F' : \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$ such that $F = U \circ F'$, the structure of an Abelian group object on $F'(X)$ induced by (1.5) coincides with the Abelian group structure already present on $F'(X)$. The functor F' commutes with products because $U \circ F'$ does and because U is conservative, so that F' takes (1.5) to $F'(X) \oplus F'(X) \xrightarrow{[\mathrm{id}_{F'(X)}\, \mathrm{id}_{F'(X)}]} F'(X)$, which is the multiplication map of the group structure on $F(X)$ determined by F. □

Lemma 1.4.3. *Let P_1, P_2 be two contravariant functors from $\mathcal{A}$ to $\mathbf{Ab}$ which commute with finite products, and denote by $U : \mathbf{Ab} \to \mathbf{Set}$ the forgetful functor, then the canonical map*

$$\mathbf{Cat}(\mathcal{A}, \mathbf{Ab})(P_1, P_2) \to \mathbf{Cat}(\mathcal{A}, \mathbf{Set})(U \circ P_1, U \circ P_2) \tag{1.6}$$

[2]Note that being an additive category is a property, not additional structure.

is a bijection. □

Proof. Because $U : \mathbf{Ab} \to \mathbf{Set}$ is faithful, the map (1.6) is injective. To see that it is surjective, let η be a natural transformation $U \circ P_1 \Rightarrow U \circ P_2$. Showing that for every object $X \in \mathcal{A}$ the map $\eta_X : U \circ P_1(X) \to U \circ P_2(X)$ is a homomorphism of Abelian groups is the same as showing that for every object $X \in \mathcal{A}$ the diagram

$$\begin{array}{ccc} U \circ P_1(X) \times U \circ P_1(X) & \xrightarrow{\eta_X \times \eta_X} & U \circ P_2(X) \times U \circ P_2(X) \\ \downarrow & & \downarrow \\ U \circ P_1(X) & \xrightarrow{\eta_X} & U \circ P_2(X) \end{array}$$

commutes, where the vertical maps are the multiplication maps. But, as both $U \circ P_1$ and $U \circ P_2$ commute with finite products, the above diagram commutes by naturality. □

Putting these lemmas together we immediately see that the Abelian category of additive contravariant functors from $\mathcal{A}$ to $\mathbf{Ab}$ is canonically isomorphic to the category of presheaves on $\mathcal{A}$ which take finite coproducts to finite products; we denote this category by $\widehat{\mathcal{A}}_{\text{add}}$. We thus have canonical embeddings

$$\mathcal{A} \hookrightarrow \text{Ind}(\mathcal{A}) \hookrightarrow \widehat{\mathcal{A}}_{\text{add}}.$$

Theorem 1.4.4. *The category* Ind($\mathcal{A}$) *is Abelian.*

Proof. By the explanation in the introduction of this section we only have to show that Ind($\mathcal{A}$) satisfies AB 2) (see [Gro57, §1.3]); this is shown in [KS06, Th. 8.6.5]. □

Remark 1.4.5. Let A be a commutative ring, then the category $\mathbf{Mod}_A^{\text{fp}}$ of finitely presented A-modules is Abelian iff A is Noetherian, however $\text{Ind}(\mathbf{Mod}_A^{\text{fp}}) \cong \mathbf{Mod}_A$ is always Abelian (see Example 1.2.2), so we have seen another way in which the ind-construction may "improve" a category. This idea is explored in detail in [Sch12]. ⌟

Proposition 1.4.6. [KS06, Th. 8.6.5.vi] *If* $\mathcal{A}$ *is essentially small, then* Ind($\mathcal{A}$) *is a Grothendieck category.* □

Proposition 1.4.7. *The compact objects in* Ind($\mathcal{A}$) *are exactly the objects in* $\mathcal{A}$.

Proof. By Remark 1.1.18 the statement is true iff $\mathcal{A}$ is idempotent complete, but this is true for any Abelian category. □

1.4.1 Pro-Artinian categories

We have already seen in Corollary 1.1.11 and Proposition 1.4.6 that if $\mathcal{A}$ is essentially small, then Pro($\mathcal{A}$) is particularly nice. Here we will show that if the objects of $\mathcal{A}$ are furthermore Artinian, then the objects in Pro($\mathcal{A}$) may be written as filtered limits in a canonical way.

Definition 1.4.8. An object $X \in \mathcal{A}$ is called *Artinian* if any descending chain of subobjects stabilises, that is, if for any chain of monomorphisms

$$X \hookleftarrow X_0 \hookleftarrow X_1 \hookleftarrow \cdots$$

there exists a number $N \in \mathbb{N}$ such that for all $i > N$ the morphisms $X_i \hookleftarrow X_{i+1}$ are isomorphisms. ⌟

Definition 1.4.9. An Abelian category is called *Artinian* if it is essentially small and all its objects are Artinian. ⌟

Definition 1.4.10. An Abelian category is called *pro-Artinian* if it is equivalent to the pro-completion of an Artinian category. ⌟

Remark 1.4.11. Abelian categories equivalent to pro-Artinian categories may also be characterised intrinsically (see [DG70, §V.2.2.1]). This is not explained however in [DG70]; one can deduce this from the results in [Gab62]. ⌟

Proposition 1.4.12. [DG70, Th. V.2.3.1] *Assume that $\mathcal{A}$ is Artinian, then the Artinian objects in $\mathrm{Pro}(\mathcal{A})$ are exactly the objects in $\mathcal{A}$.* □

Theorem 1.4.13. *Assume that $\mathcal{A}$ is Artinian. Consider an object $X \in \mathrm{Pro}(\mathcal{A})$ and let $\{X_i\}$ be the directed set of all sub-objects of X such that X/X_i is compact, then the canonical morphism $X \to \varprojlim X/X_i$ is an isomorphism.*

Proof. Put together [Gab62, Prop. I.6.6.b], the discussion directly preceding [Gab62, Prop. II.3.7] and [KS06, Ex. 5.2.2.iii]. □

Chapter 2

Group Schemes

In this chapter we review the notion of group scheme, look at some basic constructions and then focus on commutative group schemes. We will devleop a more extensive theory than what is strictly required so that we may see how these facts needed later on fit into a wider picture; in particular we will witness the remarkable fact that many subcategories of the category of commutative k-schemes are Abelian.

2.1 Basic theory

2.1.1 Basic definitions and properties

Definition 2.1.1. Let S be a scheme, then a *group scheme over* S or an S*-group* is a group object in the category of S-schemes. If S is isomorphic to the spectrum of a ring k, then we also call such a group object a k-group. ⌟

We will only consider group schemes over a field, though we wish to point out that many of the concepts discussed bellow still make sense over more general base schemes (see [SGA 3_{I}], [SGA 3_{II}], [SGA 3_{III}]).
For the rest of this chapter k denotes a ring and p its characteristic. We begin by discussing two basic properties of group schemes over a field.

Proposition 2.1.2. *Assume that k is a field and let G be a k-group, then the unit morphism* $\operatorname{Spec} k \to G$ *is a closed immersion.*

Proof. Denote by e the unit element of G. The unit morphism $\operatorname{Spec} k \to \operatorname{Spec} A$ has a left inverse for every affine neighbourhood $\operatorname{Spec} A$ of e, so that the ring homomorphism $A \to k$ has a right inverse and is thus surjective. This shows that the unit morphism $\operatorname{Spec} k \to G$ restricts to a closed immersion for every affine open neighbourhood of e; as the property of being a closed immersion is local on the target (see e.g. [EGA I, Cor. 4.2.4]) the proof is complete. □

Corollary 2.1.3. *Every group scheme over a field is separated*[1].

[1] This is not true for group schemes over an arbitrary base scheme; see [SGA 3_{I}, Ex. $\mathrm{VI_B}$ 5.6.4].

Proof. Let k be a field and let G be a k-group. Denote by $\varphi : G \times_k G \to G$ the morphism given by composing $\mathrm{id}_G \times (\_)^{-1} : G \times_k G \to G \times_k G$ with the multiplication morphism of G, then it is straightforward to check that

$$\begin{array}{ccc} G & \longrightarrow & \operatorname{Spec} k \\ \big\downarrow{\scriptstyle\Delta} & & \big\downarrow \\ G \times_k G & \xrightarrow{\varphi} & G \end{array}$$

is Cartesian, where the vertical morphism to the right is the unit morphism. As the property of begin a closed immersion is stable under base change (see [GW10, Prop. 4.32.2]) we are done. □

Definition 2.1.4. A k-group is called *affine* if its underlying k-scheme is affine[2]. ⌟

Recall that a group object in a category $\mathcal{C}$ may equivalently be defined as an object $G \in \mathcal{C}$ together with a multiplication morphism $G \times G \to G$, a unit morphism $1 \to G$ and an inverse morphism $G \to G$ satisfying certain properties, or as an object $G \in \mathcal{C}$ together with a factorisation of the presheaf $\mathcal{C}(\_, G)$ through the forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ (see e.g. [Mac98, III.6]). This provides two useful perspectives, the second of which may be modified as follows in the context of k-schemes: The composition of the Yoneda embedding $\mathbf{Sch}_k \hookrightarrow \widehat{\mathbf{Sch}_k}$ with the functor $\widehat{\mathbf{Sch}_k} \to \widehat{\mathbf{Aff}\,k}$ induced by the inclusion $\mathbf{Aff}\,k \hookrightarrow \mathbf{Sch}_k$ is fully faithful. (This is discussed in great detail in [DG70, §I.1]; for a quick proof see [EH00, Prop. VI-2].) This in turn means that any k-scheme may be viewed as a copresheaf on $\mathbf{Alg}_k$. Group schemes are often easiest to describe as group valued functors on $\mathbf{Alg}_k$. For affine k-groups there is a third perspective: the anti-equivalence between affine k-schemes and k-algebras extends to an anti-equivalence between affine k-groups and Hopf algebras over k.
Apart from affine group schemes we will focus on two more types of group schemes: Algebraic and pro-algebraic group schemes.

Definition 2.1.5. If k is a field, then a group scheme over k is called (locally) *algebraic* if its underlying scheme is (locally) algebraic. ⌟

Recall that a scheme over a ring is called (locally) algebraic if its structure morphism is (locally) of finite type or of finite presentation, depending on the author. For schemes over a Noetherian ring (and in particular over a field) these two notions coincide. Some authors require morphisms of finite type/presentation to be separated (see e.g. [DG70, Def. I.3.1.6]) while others do not (see e.g. [EGA I, Def. 6.3.2]); by Corollary 2.1.3 this point is immaterial for group schemes over a field.

Proposition 2.1.6. *A locally algebraic k-group is smooth iff it is geometrically reduced. In particular, if k is perfect, then a k-group is smooth iff it is reduced.*

Proof. For the first statement see [GW10, Prop. 16.49]. The second statement follows from the fact that if k is perfect, then a k-scheme which is locally of finite type is reduced iff it is geometrically reduced (see [GW10, Cor. 5.57]). □

[2]More generally, a group scheme over an arbitrary base scheme is called affine if the structure morphism is affine.

Theorem 2.1.7 (Cartier)**.** [DG70, Th. II.6.1.1] *Every locally algebraic group scheme over a field of characteristic* 0 *is reduced; in other words, every locally algebraic group scheme over a field of characteristic* 0 *is smooth.* □

Definition 2.1.8. If k is a field, then an object of the pro-completion of the category of algebraic k-groups is called a *pro-algebraic k-group.* ⌟

2.1.2 Constructions on group schemes

The only three constructions we will be interested in are products, subgroups and quotients. Throughout this subsection k is assumed to be a field.

Products

Let $\mathcal{C}$ be a category admitting finite products, then $\mathrm{Grp}(\mathcal{C})$ admits finite products; indeed, the product of two group valued functors is given objectwise, so the underlying object of the product of two group objects is simply the product of the respective underlying objects. We thus see that products exist in the category of (affine) group schemes over any base scheme.

Subgroups

Proposition 2.1.9. *Let G and H be k-groups and let $\varphi : G \to H$ be a morphism of k-groups. If G and H are both affine or both algebraic, then the following three properties are equivalent:*

(I) *The morphism $\varphi : G \to H$ is a monomorphism in the category of k-groups.*

(II) *The morphism $\varphi : G \to H$ is an immersion.*

(III) *The morphism $\varphi : G \to H$ is a closed immersion.*

If G and H are affine, the properties (I) - (III) *are furthermore equivalent to*

(IV) *The homomorphism of Hopf algebras $\mathscr{O}(\varphi) : \mathscr{O}(H) \to \mathscr{O}(G)$ is surjective.*

Proof. In both cases the implications (III) $\implies$ (II) and (II) $\implies$ (I) are clear. The proof of (I) $\implies$ (III) may be found in [DG70, Th. III.3.7.2.b] for affine k-groups, and in [DG70, Prop. II.5.5.1.b] for algebraic k-groups. The proof of (I) $\iff$ (IV) may be found in [DG70, Th. III.3.7.2.b]. □

Definition 2.1.10. A *k-subgroup*, or simply *subgroup*, is a morphism of affine or algebraic k-groups satisfying the equivalent properties in Proposition 2.1.9. ⌟

Convention 2.1.11. Let $H \hookrightarrow G$ be a k-subgroup, then we will often refer to H as a k-subgroup of G. ⌟

Proposition 2.1.12. *Any k-subgroup of an algebraic k-group is algebraic and any k-subgroup of an affine k-group is affine.* □

Corollary 2.1.13. *Subgroups of commutative affine or algebraic k-groups are commutative.*

Proof. Let $G \hookrightarrow H$ be a subgroup, then for every k-algebra A the map $G(A) \to H(A)$ is a monomorphism, so that $G(A)$ has the structure of an Abelian group. □

Quotients

Definition 2.1.14. Let G be a k-group and X a k-scheme, then a morphism $G \times X \to X$ is called a *(left) action of G on X* if for every k-scheme S the map $G(S) \times X(S) \to X(S)$ is a (left) action of $G(S)$ on $X(S)$. ⌟

Definition 2.1.15. Let G be a k-group, let X be a k-scheme, and let $\alpha : G \times X \to X$ be an action of G on X. If it exists, the coequaliser of the diagram

$$G \times X \underset{\alpha}{\overset{p}{\rightrightarrows}} X,$$

where p denotes the projection morphism, is called the *quotient of X by α*, or if α is understood from context, the *quotient of X by G*. ⌟

Let $H \hookrightarrow G$ be a k-subgroup, then there is a canonical action $H \times G \to G$ given by $(h, g) \mapsto hg$. If it exists we will thus speak of the quotient of G by H.
In certain cases the existence of the quotient of a k-scheme by a k-group may be proved by giving a rather explicit construction (see e.g. [DG70, Cor. III.2.6.1]), however, in the situation we are interested in, that is, quotients by subgroups, the most common approach is to work with fppf sheaves. As the fppf topology is subcanonical, any k-group may be viewed as a (group-valued) fppf sheaf and any k-subgroup may then be viewed as a subsheaf. Let G be a k-group acting on a k-scheme X, then the quotient sheaf exists; this is seen by first forming the quotient presheaf (which exists because quotients of group actions exist in the category of sets), and then applying the sheafification functor, which is exact. One then tries to prove that this quotient sheaf is representable.

Theorem 2.1.16. *Let G be a k-group and H a subgroup, then the quotient of G by H exists if both groups are affine or if both groups are locally algebraic; these k-groups are again affine and locally algebraic respectively.*

Proof. The first assertion is proved in both [SGA 3_{I}, Th. VI_{B}.11.17] and [DG70, Th. III.3.7.2], and the second assertion is proved in [SGA 3_{I}, Th. VI_{A}.3.2][3]. □

Proposition 2.1.17. [DG70, Th.III.3.7.2] *A morphism of affine k-groups is a quotient iff the corresponding homomorphism of Hopf k-algebras is injective iff it is faithfully flat.* □

Proposition 2.1.18. *The quotient of a commutative algebraic or affine k-group is commutative.*

Proof. Let G be a commutative algebraic or affine k-group and let H be a k-subgroup, then the copresheaf on $\mathbf{Alg}_k$ given by $A \mapsto G(A)/H(A)$ is clearly commutative; as sheafification is exact it takes commutative group objects to commutative group objects, so that the associated fppf-sheaf is commutative, and thus also the k-group which represents it. □

[3] In fact the statement is proved more generally for flat, locally algebraic groups over any Artinian ring.

2.1.3 Examples

Example 2.1.19. The presheaf of groups

$$\begin{array}{rcl} \mathbf{Sch}_k^{\mathrm{op}} & \to & \mathbf{Grp} \\ X & \mapsto & \mathscr{O}(X)^+ \end{array}$$

(where A^+ denotes the underlying Abelian group of any ring A) is represented by $\mathbb{A}^1_k$; this k-group is called the *additive k-group* and is denoted by $\mathbb{G}_{k,a}$, or simply $\mathbb{G}_a$ when no confusion arises. The corresponding copresheaf of groups on $\mathbf{Alg}_k$ is then given by

$$\begin{array}{rcl} \mathbf{Alg}_k & \to & \mathbf{Grp} \\ A & \mapsto & A^+. \end{array}$$

As it is particularly neat, we give the corresponding cogroup structure on $k[X]$:

$$\begin{array}{rcll} k[X] & \to & k[X_1, X_2] & \text{(multiplication)} \\ X & \mapsto & X_1 + X_2 & \\ & & & \\ k[X] & \to & k & \text{(unit)} \\ X & \mapsto & 0 & \\ & & & \\ k[X] & \to & k[X] & \text{(inverse)} \\ X & \mapsto & -X. & \end{array}$$

It is interesting to not only consider the presheaf represented by $\mathbb{G}_a$ on $\mathbf{Sch}_k$ but also on $\mathrm{Grp}(\mathbf{Sch}_k)$. The functor $\mathrm{Grp}(\mathbf{Sch}_k)(\_, \mathbb{G}_a) : \mathrm{Grp}(\mathbf{Sch}_k)^{\mathrm{op}} \to \mathbf{Grp}$ factors through the forgetful functor $\mathbf{Vect}_k \to \mathbf{Grp}$, as it assigns to each affine k-group G the k-submodule of $\mathscr{O}(G)$ given by

$$\left\{ f \in \mathscr{O}(G) \;\middle|\; \Delta(f) = 1 \otimes f + f \otimes 1 \right\}.$$

⌟

Example 2.1.20. The presheaf of groups

$$\begin{array}{rcl} \mathbf{Sch}_k^{\mathrm{op}} & \to & \mathbf{Grp} \\ X & \mapsto & \mathscr{O}(X)^\times \end{array}$$

is represented by $\mathbb{A}^1_k \setminus \{0\}$; this k-group is called the *multiplicative k-group* and is denoted by $\mathbb{G}_{k,m}$, or simply $\mathbb{G}_m$ when no confusion arises. The corresponding copresheaf of $\mathbf{Alg}_k$ is then given by

$$\begin{array}{rcl} \mathbf{Alg}_k & \to & \mathbf{Grp} \\ A & \mapsto & A^\times. \end{array}$$

⌟

Example 2.1.21. Let G be a discrete topological group, and assume that k is Artinian, then the presheaf of groups

$$\begin{array}{rcl} \mathbf{Sch}_k^{\mathrm{op}} & \to & \mathbf{Grp} \\ (X, \mathscr{O}_X) & \mapsto & \mathbf{Top}(X, G) \end{array}$$

is represented by $\coprod_{g \in G} \operatorname{Spec} k$. Such groups are called *constant k-groups.* ⌟

Example 2.1.22. Assume that k is a field, then the copresheaf of groups

$$\begin{array}{rcl} \mathbf{Alg}_k & \to & \mathbf{Grp} \\ A & \mapsto & \mathrm{GL}_n(A) \end{array}$$

is represented by $k[X_{ij}]_{\det[X_{ij}]}$; the corresponding k-group is called the *general linear k-group of degree n* and is denoted by $\mathrm{GL}_{n,k}$ or simply by GL_n when this causes no confusion. ⌟

Affine algebraic k-groups are sometimes called linear k-groups due to the following proposition.

Proposition 2.1.23. [DG70, Cor. II.5.5.2] *An algebraic k-group G is affine iff there exists a positive integer $n \in \mathbb{N}$ and a monomorphism $G \hookrightarrow \mathrm{GL}_n$.* □

Example 2.1.24. The copresheaf of groups

$$\begin{array}{rcl} \mathbf{Alg}_k & \to & \mathbf{Grp} \\ A & \mapsto & \mathrm{SL}_n(A) \end{array}$$

is represented by $k[X_{ij}]/(\det[X_{ij}] - 1)$; the corresponding k-group is called the *special linear k-group of degree n* and is denoted by $\mathrm{SL}_{n,k}$ or simply by SL_n when this causes no confusion. ⌟

Example 2.1.25. The subcopresheaf of GL_n given by

$$\begin{array}{rcl} \mathbf{Alg}_k & \to & \mathbf{Grp} \\ A & \mapsto & \left\{ (a_{ij}) \in \mathrm{GL}_n(A) \,\middle|\, a_{ij} = 0 \text{ for } i > j \text{ and } a_{ij} = 1 \text{ for } i = j \right\} \end{array}$$

is representable by a subgroup of GL_n which is denoted by $\mathbb{U}_{n,k}$ or simply by $\mathbb{U}_n$ if k is clear from context. ⌟

Example 2.1.26. The kernel of the morphism $\mathbb{G}_m \to \mathbb{G}_m$ determined by

$$\begin{array}{rcl} k[X, X^{-1}] & \to & k[X, X^{-1}] \\ X & \mapsto & X^n \end{array}$$

corresponds to the copresheaf of groups

$$\begin{array}{rcl} \mathbf{Alg}_k & \to & \mathbf{Grp} \\ A & \mapsto & \left\{ x \in A \,\middle|\, x^n = 1 \right\} \end{array}$$

and is represented by $k[X]/(X^n - 1)$. This k-group is called the *k-group of the n-th roots of unity* and is denoted by $\mu_{n,k}$ or simply by μ_n when this causes no confusion. ⌟

Example 2.1.27. If $p > 0$, then for every $r \in \mathbb{N}_{>0}$ the kernel of the morphism $\mathbb{G}_a \to \mathbb{G}_a$ determined by

$$\begin{array}{rcl} k[X] & \to & k[X] \\ X & \mapsto & X^{p^r} \end{array}$$

corresponds to the copresheaf of groups

$$\begin{array}{rcl} \mathbf{Alg}_k & \to & \mathbf{Grp} \\ A & \mapsto & \left\{ \, x \in A \;\middle|\; x^{p^r} = 0 \, \right\} \end{array}$$

and is represented by $k[X]/X^{p^r}$. This k-group is denoted by $\alpha_{p^r,k}$ or simply by α_{p^r} when this causes no confusion. ⌟

Example 2.1.28. Any elliptic curve over k is an example of a commutative k-group. ⌟

2.2 Categories of group schemes

In this section we assume that k is a field. We now start concentrating on commutative k-groups, and we will observe the remarkable fact that many subcategories of the category of commutative k-groups are Abelian.

Theorem 2.2.1. [SGA 3_{I}, Th. VI_{A}.5.4.2] *The category of commutative algebraic k-groups is Abelian.* □

We denote this category by $\mathbf{AGS}_k$. By Theorem 1.4.4 we obtain the following corollary.

Corollary 2.2.2. *The category of commutative pro-algebraic k-groups is Abelian.* □

We denote this category by $\mathbf{PAGS}_k$.
It may be shown that the full subcategory or $\mathbf{AGS}_k$ of commutative affine algebraic k-groups is closed under kernels, cokernels and extension so that we obtain the following theorem.

Theorem 2.2.3. [SGA 3_{I}, Cor. 5.4.3] *The category of affine commutative algebraic k-groups is Abelian.* □

We denote this category by $\mathbf{AAGS}_k$.

Remark 2.2.4. The fact that $\mathbf{AAGS}_k$ is Abelian can be proved purely in the setting of Hopf-algebras (see [Tak72, Th. 3.1]). ⌟

Theorem 2.2.5. *The pro-completion of the category of affine algebraic k-groups is equivalent to the category of affine k-groups.*

Proof. This follows immediately from Example 1.3.7 and the anti-equivalence between the category of affine k-groups and the category of Hopf k-algebras. □

Remark 2.2.6. As every Hopf k-algebra is the filtered colimit of all its finitely presented Hopf k-subalgebras, we see by Proposition 2.1.17 that every affine k-group is the filtered limit of all its algebraic quotients. (For commutative affine k-groups this can also be shown more abstractly using Theorem 1.4.13.) ⌟

Theorem 2.2.7. *The pro-completion of the category of commutative affine algebraic k-groups is equivalent to the category of commutative affine k-groups.*

Proof. This follows immediately from a slight modification of Example 1.3.7 by replacing the expression "cogroup object" with the expression "cocommutative cogroup object" and the anti-equivalence between the category of commutative affine k-groups and the category of cocommutative Hopf k-algebras. □

Corollary 2.2.8. *The category of commutative affine k-groups is Abelian.*

Proof. This follows immediately from the preceding theorem and Theorem 1.4.4. □

Remark 2.2.9. This is proved directly in [DG70, Cor. III.3.7.4]. Considering also Remark 2.2.4 we see that there are thus three distinct ways of proving this fact. ⌟

We denote this category by $\mathbf{PAAGS}_k$.

2.3 Structure theory of commutative group schemes

In this section we assume that k is a field. Algebraic k-groups and commutative affine k-groups possess a well-understood, elegant structure theory (for the former see [Mil12, p. 15], the latter will be explained in this chapter). As in §2.1 we will present the theory in a wider context than we need.

We begin with a quick overview before moving on to the details which are presented in the following subsections: Most importantly, any locally algebraic k-group is the extension of an étale k-group (see §2.3.1) by a connected algebraic k-group. This constitutes the algebraic analogue of the fact that any Lie group is the extension of a discrete group by the connected component of its unit. If k is perfect, then any smooth connected k-group is the extension of an Abelian k-variety (see §2.3.2) by an affine algebraic k-group. Finally, any commutative affine k-group is the extension of a unipotent k-group (see §2.3.5) by a k-group of multiplicative type (see §2.3.4).

2.3.1 Étale group schemes

In this section we consider a generalisation of constant group schemes (see Example 2.1.21).

Definition 2.3.1. A k-group is called *étale* if its underlying k-scheme is étale, that is, if its structure morphism is étale. ⌟

Proposition 2.3.2. [DG70, Prop. I.4.6.1] *A k-scheme is étale iff it is the disjoint union of affine k-schemes of the form* $\operatorname{Spec} k'$*, where* $k \hookrightarrow k'$ *is a finite, separable field extension.* □

Corollary 2.3.3. *Every constant k-scheme is étale. The converse is true if k is separably closed.* □

Corollary 2.3.4. *An étale k-scheme is algebraic iff it is finite iff it is affine.* □

Proof. If an étale k-scheme is affine, then it is isomorphic to the spectrum of an étale k-algebra, so it is finite and a fortiori algebraic. On the other hand, if an étale k-scheme is algebraic, then it is in particular quasi-compact, so that it is isomorphic to the finite coproduct of k-schemes of the form $\operatorname{Spec} k'$, where $k \hookrightarrow k'$ is a finite separable field extension, so it must be the affine k-scheme of the product of separable field extensions of k. □

The most important example of étale k-groups for us will be the following.

Example 2.3.5. [DG70, Ex. I.5.1.5] For every $n > 0$ the k-group of the n-th roots of unity μ_n (see Example 2.1.26) is étale iff $n \cdot 1_k \neq 0$. ⌟

We now give the precise result stating how étale k-groups fit into the structure theory of locally algebraic k-groups.

Theorem 2.3.6. [DG70, Th. II.5.1.1, Prop. II.5.1.8] *Let G be a locally algebraic k-group, then the connected components of G are irreducible, algebraic and of the same dimension. The connected component G° of the unit $e \in G(k)$ is a k-subgroup. The k-group G fits into a short exact sequence*

$$0 \to G^\circ \to G \to \pi_0(G) \to 0,$$

where $\pi_0(G)$ is an étale k-group. The fibres of $G \to \pi_0(G)$ are the connected components of G. □

Definition 2.3.7. Let G be as in Theorem 2.3.6, then G° is called the *identity component of G* and $\pi_0(G)$ is called the *k-group of connected components of G*. ⌟

Corollary 2.3.8. *A locally algebraic k-group G is algebraic iff $\pi_0(G)$ is finite.* □

Corollary 2.3.9. *The map sending any locally algebraic k-group to its k-group of connected components extends to a functor.* □

Proof. This follows from the fact that for any locally algebraic k-groups G and H with G connected any morphism $\varphi : G \to H$ must factor through $H^\circ \hookrightarrow H$. □

Corollary 2.3.10. *The functor π_0 preserves epimorphisms.* □

Proposition 2.3.11. [DG70, Cor. I.4.6.8.iii] *A locally algebraic k-group G is connected iff $\pi_0(G) \cong \operatorname{Spec} k$.* □

Proposition 2.3.12. [DG70, Cor. I.4.6.10] *The functor π_0 preserves finite products.* □

Corollary 2.3.13. *The product of two locally algebraic k-groups is connected iff its factors are connected.*

Proof. Let G_1, G_2 be locally algebraic k-groups. The k-group $\pi_0(G_1 \times_k G_2) \cong \pi_0(G_1) \times_k \pi_0(G_2)$ is the disjoint union of the k-groups $\operatorname{Spec} k_1 \times_k \operatorname{Spec} k_2$, $\operatorname{Spec} k_1$ and $\operatorname{Spec} k_2$ are k-subschemes of $\pi_0(G_1)$ and $\pi_0(G_2)$ respectively; thus the number of connected components of $\pi_0(G_1 \times_k G_2)$ is greater or equal to the number of connected components of G_1 times the number of connected components of G_2. It is thus enough to check that if both G_1 and G_2 are connected, then we have $\pi_0(G_1 \times_k G_2) \cong \operatorname{Spec} k \times_k \operatorname{Spec} k \cong \operatorname{Spec} k$. □

The following corollary of Theorem 2.3.6 is needed for the subsequent lemma.

Corollary 2.3.14. *A locally algebraic k-group is connected iff it is irreducible.* □

We will need the following lemma in Proposition 2.3.36.

Lemma 2.3.15. *An affine k-group is connected iff it is irreducible.*

Proof. That any irreducible affine k-group is connected is clear. To prove the converse, let G be a connected affine k-group, then all its affine algebraic quotients are connected an therefore irreducible. Assume there existed two non-zero elements x, y in the corresponding Hopf k-algebra $\mathscr{O}(G)$ such that $xy = 0$, then these would be contained in a finitely generated Hopf subalgebra of $\mathscr{O}(G)$, which leeds to a contradiction (see Example 1.3.7). □

We finish this subsection with a description of the category of commutative, finite étale k-groups.

Proposition 2.3.16. *The category of commutative, finite étale k-groups forms an Abelian subcategory of* $\mathbf{PAAGS}_k$.

Proof. It is easily checked that the product of two étale algebras is again étale, so that the category of commutative, finite étale k-groups is closed under coproducts, and is thus additive. By [Bou70, Prop. V.6.4.3] subalgebras and quotients of étale algebras are again étale algebras, so that the category of commutative, finite étale k-groups is also closed under kernels and cokernels (see Propositions 2.1.9 and 2.1.17). □

2.3.2 Abelian varieties

Definition 2.3.17. A k-group is called an *Abelian k-variety* or *Abelian variety over k* if it is smooth, connected and proper. ⌟

Example 2.3.18. Elliptic curves over k are exactly the Abelian k-varieties of dimension 1 (see [Hid13, Lm 6.1]). ⌟

Just as for elliptic curves, general Abelian varieties are commutative, and the group structure is uniquely determined by the choice of a zero element.

Proposition 2.3.19. [Con02, Lm. 2.2] *Let A be an Abelian k-variety, G a connected smooth k-group, and let $\varphi : G \to A$ be a morphism of k-schemes mapping the unit element of G to the unit element of A, then φ is a morphism of k-group schemes.* □

Corollary 2.3.20. *Let A be an Abelian variety over k, then*

(I) *any group structure on the underlying k-scheme of A with the same zero element coincides with the given group structure;*

(II) *the k-group A is commutative.*

Proof. (I) Denote by A' an Abelian variety with the same underlying k-scheme and zero element as A, then the identity morphism of k-schemes of the underlying scheme of A and A' induces an isomorphism between A and A'.
(II) The inverse image morphism $\iota : A \to A$ is a k-group morphism; this property uniquely characterises commutative group objects in any category. □

Remark 2.3.21. Just like for elliptic curves there is an intrinsic way to define the group law of any Abelian variety (see [MO25262]). ⌟

The following theorem is important to characterise sheaves associated to Abelian varieties (see §2.4).

Theorem 2.3.22. [BLR90, Cor. 8.4.6] *Let A be an Abelian k-variety and X a regular k-scheme, then any rational k-morphism $X \dashrightarrow A$ is defined everywhere.* □

Theorem 2.3.23. [CS86, Th. V.7.1] *Every Abelian variety is projective.* □

Theorem 2.3.24 (Chevalley)**.** [Con02, Lm. 2.2] *Assume that k is perfect, then for any smooth, connected, algebraic k-group G there exists a unique short exact sequence of smooth, connected, algebraic k-groups*

$$0 \to H \to G \to A \to 0$$

such that H is affine and A is an Abelian k-variety. □

We thus see that if k is perfect, then every smooth, connected, algebraic k-group is the extension of a projective k-group by an affine k-group.
Finally, turning to the category of Abelian k-varieties, we see that it forms an additive subcategory of $\mathbf{AGS}_k$, i.e. it is closed under products, but is not an Abelian subcategory as the kernel of a morphism of Abelian k-varieties (in $\mathbf{AGS}_k$) may be discrete, and thus not connected (see [CS86, Th. V.8.2]). Of course this does not a priori exclude the possibility that the category of Abelian k-varieties forms an Abelian category, but we are unaware whether this is the case.

2.3.3 Finite group schemes

We now make a brief diversion from the structure theory outlined in the introduction of this section. We have seen in Theorem 2.3.24 that every algebraic k-group may be decomposed into an affine and a projective part. In this subsection we briefly study those algebraic k-groups which are both affine and projective; these will play an important role in the proof of Theorem 5.1.1.

Proposition 2.3.25. *A k-group is finite iff it is both affine and projective.*

Proof. This is simply a corollary of the fact that a morphism of schemes is finite iff it is affine and projective (see [GW10, Cor. 13.77]). □

We have see in Corollary 2.3.3 that if k is separably closed, then the notions of constant and étale k-groups coincide. We now expand this result to the case of algebraic k-groups.

Proposition 2.3.26. *If k is separably closed, then an algebraic k-group is finite iff it is étale iff it is constant.*

Proof. We have already proved the second equivalence in Corollary 2.3.3. For the first equivalence see [Pin05, Prop. 12.1]. □

Theorem 2.3.27. [Pin05, Th. 10.5] *The category of commutative finite k-groups is Abelian.* □

Remark 2.3.28. This may be proved purely in the setting of Hopf k-algebras (see [Swe69, Ch. XVI]). ⌟

2.3.4 Groups of multiplicative type

Groups of multiplicative type are defined in terms of diagonalisable groups, so we study these first.

Diagonalisable k-groups

Consider the functor

$$\begin{array}{rcl} D : \mathbf{Ab} & \to & \mathbf{Cat}(\mathbf{Alg}_k, \mathbf{Set}) \\ \Gamma & \mapsto & (A \mapsto \mathbf{Ab}(\Gamma, A^\times)). \end{array}$$

For every Abelian group Γ the functor $D(\Gamma)$ is representable by $k[\Gamma]$, the group algebra of Γ over k, so that it is a commutative, affine k-group.

Definition 2.3.29. A k-group is called *diagonalisable* if it lies in the essential image of D. ⌟

Let Γ be an Abelian group, then there is a canonical isomorphism $k[\Gamma] \otimes k[\Gamma] \cong k[\Gamma \times \Gamma]$ owing to the fact that the underlying k-module of $k[\Gamma]$ is simply a k-vector space with basis Γ, so that the underlying k-module of $k[\Gamma] \otimes k[\Gamma]$ must be canonically isomorphic to the k-vector space with basis $\Gamma \times \Gamma$; it is then straightforward to check that the induced homomorphism of k-modules is compatible with multiplication. It is now not hard to see that the Hopf k-algebra structure on $k[\Gamma]$ is given by

$$\begin{array}{rcll} k[\Gamma] & \to & k[\Gamma \times \Gamma] & \text{(multiplication)} \\ g & \mapsto & (g, g) & \\ & & & \\ k[\Gamma] & \to & k & \text{(unit)} \\ g & \mapsto & 1 & \\ & & & \\ k[\Gamma] & \to & k[\Gamma] & \text{(inverse)} \\ g & \mapsto & -g. & \end{array}$$

Theorem 2.3.30. [DG70, Prop. II.1.2.11 & Cor. IV.1.1.3] *The functor $D : \mathbf{Ab} \to \mathbf{PAAGS}_k$ is fully faithful and exact.* □

Corollary 2.3.31. *The category of diagonalisable k-groups is Abelian.* □

Proposition 2.3.32. *Let Γ be an Abelian group, then Γ is finitely generated iff $D(\Gamma)$ is algebraic.* □

Proof. Let Γ be an Abelian group and assume that it is generated by the elements $g_1, \dots, g_n$, then it is clear that these same elements generate $D(\Gamma)$. Conversely, assume that $D(\Gamma)$ is algebraic, then there are elements $\sum_{i=1}^{n_0} x_{0\,i}\, g_{0\,i}, \dots, \sum_{i=1}^{n_N} x_{N\,i}\, g_{N\,i}$, where all $x_{i\,j} \in k$ and $g_{i\,j} \in \Gamma$, which generate $k[\Gamma]$. We claim that the set $E := \{g_{0\,1}, g_{0\,2}, \dots, g_{N\,n_N-1}, g_{N\,n_N}\}$ generates Γ. Any element $g \in \Gamma$ may be written as $\sum_{i=1}^{m} x_i g'_i$ with all $x_i \in k$ and g'_i being a product of elements in E, then, as Γ forms a basis of the underlying vector space of $k[\Gamma]$, we have $x_i = 1$ for some $i \in \{1, \dots, m\}$ and $x_j g'_j = 0$ for those $j \in \{1, \dots, m\}$ such that $i \neq j$. □

Corollary 2.3.33. *The category of finitely generated diagonalisable k-groups is Abelian.* □

Proposition 2.3.34. [DG70, Cor. IV.1.1.7] *The category of diagonalisable k-groups is closed under all limits and all finite colimits in $\mathbf{PAGS}_k$.*

Corollary 2.3.35. *The category of diagonalisable k-groups is equivalent to the pro-completion of the category of algebraic diagonalisable k-groups.*

Proof. Algebraic diagonalisable k-groups are compact in the category of diagonalisable k-groups so that by Remark 2.2.6 we see that every diagonalisable k-group is the filtered colimit of algebraic diagonalisable k-groups. The statement then follows by Corollary 1.1.17. □

Let Γ be a finitely generated Abelian group, then

$$\Gamma \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}/p_1^{r_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_n^{r_n}\mathbb{Z},$$

where $p_1, \dots, p_n$ are prime numbers, and $r_1, \dots, r_n$ are non-negative integers. We thus see that

$$D(\Gamma) \cong \mathbb{G}_m \oplus \cdots \oplus \mathbb{G}_m \oplus \mu_{p_1^{r_1}} \oplus \cdots \oplus \mu_{p_n^{r_n}}.$$

W.l.o.g. we assume that for some $m \in \mathbb{N}$ we have $p_1 = \cdots = p_m = p$ and that $p_{m+1}, \dots, p_n$ are unequal to p (thus m is necessarily 0 if $p = 0$), then the short exact sequence

$$0 \to \mathbb{G}_m \oplus \cdots \oplus \mathbb{G}_m \oplus \mu_{p_1^{r_1}} \oplus \cdots \oplus \mu_{p_m^{r_m}} \to D(\Gamma) \to \mu_{p_{m+1}^{r_{m+1}}} \oplus \cdots \oplus \mu_{p_n^{r_n}} \to 0$$

is the one from Theorem 2.3.6, where $\mathbb{G}_m \oplus \cdots \oplus \mathbb{G}_m \oplus \mu_{p_1^{r_1}} \oplus \cdots \oplus \mu_{p_m^{r_m}}$ is isomorphic to the identity component of $D(\Gamma)$ and $\mu_{p_{m+1}^{r_{m+1}}} \oplus \cdots \oplus \mu_{p_n^{r_n}}$ is isomorphic to the k-group of connected components of $D(\Gamma)$.

Proposition 2.3.36. *Let Γ be an Abelian group.*

(1) *If $p = 0$, then $D(\Gamma)$ is reduced. If $p > 0$, then $D(\Gamma)$ is reduced iff the p-primary subgroup of Γ is zero.*

(2) *If $p = 0$, then $D(\Gamma)$ is connected iff Γ has no torsion. If $p > 0$, then $D(\Gamma)$ is connected iff the torsion subgroup of Γ is equal to its p-primary subgroup.*

Proof. (1) To see whether an element $\sum_{i=1}^{n} x_i g_i \in k[\Gamma]$ with $x_i \in k$ and $g_i \in \Gamma$ for all $i \in \{1, \ldots, n\}$ is nilpotent, it is enough to consider the subgroup of Γ generated by the elements $g_1, \ldots, g_n$, so we may assume that Γ is finitely generated. If $p = 0$, then $D(\Gamma)$ is reduced as by the preceding discussion $D(\Gamma)$ is the product of finitely many finite étale k-groups and copies of $\mathbb{G}_m$, which are all geometrically reduced; alternatively one may simply invoke Proposition 2.1.6 and Theorem 2.1.7. We now assume that $p > 0$. If for none of the factors $\mu_{q^r} \subseteq D(\Gamma)$ we have that $q = p$, then as for the case $p = 0$ the k-group $D(\Gamma)$ is the product of geometrically reduced k-groups. Conversely, assume $q = p$ and denote by $g \in \Gamma$ the generator of the corresponding cyclic group, then $g - 1 \in k[\Gamma]$ is nilpotent.
(2) By Lemma 2.3.15 $D(\Gamma)$ is connected iff it is irreducible; to check irreducibility we may assume, like in the previous point, that $k[\Gamma]$ and thus Γ is finitely generated (see Proposition 2.3.32). We denote by $T(\Gamma)$ the torsion subgroup of Γ. If $p = 0$, then $D(\Gamma)$ is connected iff its k-group of connected components $D(T(\Gamma))$ is isomorphic to zero (see the preceding discussion). Assume then that $p > 0$, then again by the preceding discussion the k-group of connected components of $D(\Gamma)$ is isomorphic to the direct sum of those subgroups $\mu_{q^r} \subseteq D(\Gamma)$ such that $q \neq p$. ☐

Proposition 2.3.37. *Let G be a diagonalisable k-group, then $\mathbf{PAAGS}_k(G, \mathbb{G}_a) \cong 0$.*

Proof. Let Γ be an Abelian group such that $G \cong k[\Gamma]$, then an element $\sum_{g\in\Gamma} x_g\, g \in k[\Gamma]$ is in $\mathbf{PAAGS}_k(G, \mathbb{G}_a)$ iff $\sum_{g\in\Gamma} x_g\, g \otimes g = \sum_{g\in\Gamma} x_g (g \otimes 1 + 1 \otimes g)$ which is only true for $\sum_{g\in\Gamma} x_g\, g = 0$ (see Example 2.1.19). ☐

Groups of multiplicative type

Theorem 2.3.38. *Let G be a k-group, then the following are equivalent:*

- (I) *There exists a field extension K of k such that $G \otimes_k K$ is diagonalisable.*
- (II) *The group $G \otimes_k \overline{k}$ is diagonalisable.*
- (III) *The group $G \otimes_k k^{\mathrm{sep}}$ is diagonalisable.*
- (IV) *The group G is affine and commutative, and the only morphism from G to $\mathbb{G}_a$ is the zero morphism.*

Proof. The implications (III) $\implies$ (II) $\implies$ (I) are clear. We now show (I) $\implies$ (IV). Let K be a field extension of k such that $G \otimes_k K$ is diagonalisable. By [DG70, Cor. I.2.3.9] and Proposition 2.1.17 $G \otimes_k K$ is affine and commutative. In Example 2.1.19 we saw that $\mathbf{PAAGS}_k(G, \mathbb{G}_{a,k})$ may be viewed as a k-submodule of $\mathscr{O}(G)$. Both $\mathbf{PAAGS}_K(G \otimes_k K, \mathbb{G}_{a,K})$ and $\mathbf{PAAGS}_k(G, \mathbb{G}_{a,k}) \otimes_k K$ may be viewed as submodules of $\mathscr{O}(G) \otimes_k K$ and it is then easily checked that $\mathbf{PAAGS}_k(G, \mathbb{G}_{a,k}) \otimes_k K \subseteq \mathbf{PAAGS}_K(G, \mathbb{G}_{a,K})$. As $\mathbf{PAAGS}_K(G, \mathbb{G}_{a,K}) \cong 0$ (see Proposition 2.3.37) we see that $\mathbf{PAAGS}_k(G, \mathbb{G}_{a,k}) \otimes_k K \cong 0$, so that also $\mathbf{PAAGS}_k(G, \mathbb{G}_{a,k}) \cong 0$.
For the implications (IV) $\implies$ (II) and (II) $\implies$ (III) we refer to [DG70, Th. IV.1.2.2] and [DG70, Cor. IV.1.3.5.] respectively. ☐

Definition 2.3.39. A k-group is said to be of *multiplicative type* if it satisfies the equivalent conditions in Theorem 2.3.38. ⌟

Unfortunately we do not know of any decomposition theorems of multiplicative groups into simpler groups such as $\mathbb{G}_m$; this will be the main reason why we will be working over a separably closed field in Part II.
We finish this subsection with some results on the category of k-groups of multiplicative type.

Proposition 2.3.40.

(1) *Every k-subgroup and every quotient of a k-group of multiplicative type is of multiplicative type.*

(2) *All limits and all finite colimits of k-groups of multiplicative type are of multiplicative type.*

Proof. By Theorem 2.3.38 we must simply show that base change by $\overline{k}$ (or k^{sep}) sends the structures in the statements of the proposition to the corresponding structures of diagonalisable k-groups. As limits commute with limits, base change commutes with limits and thus also monomorphisms, so we have proved the first respective parts of the two statements (see Proposition 2.1.9). The property "faithfully flat" is stable under base change, which proves the second part of (1) (see Proposition 2.1.17). For the final part of (2) we may simply note that as $\mathbf{PAAGS}_k$ is Abelian, finite coproducts coincide with products. □

Corollary 2.3.41. *The category of k-groups of multiplicative type is equivalent to the pro-completion of the category of algebraic k-groups of multiplicative type.*

Proof. The proof is completely analogous to the proof of Corollary 2.3.35. □

Corollary 2.3.42. *The categories of k-groups of multiplicative type and of algebraic k-groups of multiplicative type both form Abelian subcategories of* $\mathbf{PAAGS}_k$. □

2.3.5 Unipotent group schemes

Definition 2.3.43. A k-group G is called *unipotent* if it is affine and if for every non-zero closed k-subgroup $H \subseteq G$ there exists a non-zero morphism $H \to \mathbb{G}_a$. ⌟

While k-groups of multiplicative type are affine k-groups with no morphism to $\mathbb{G}_a$, unipotent k-groups are required to have such morphisms.

Proposition 2.3.44. [DG70, Prop. IV.2.2.5] *Let G be an affine, algebraic k-group, then the following are equivalent:*

(I) *G is unipotent.*

(II) *There exists an $n \in \mathbb{N}$ and a monomorphism $G \hookrightarrow \mathbb{U}_n$ (see Example 2.1.25).*

(III) *G possesses a composition series whose quotients are isomorphic to $\mathbb{G}_a$ if $\operatorname{char} k = 0$, and whose quotients are isomorphic to $\mathbb{G}_a$, α_p, or a finite étale k-subgroup of $\mathbb{G}_a$ if $\operatorname{char} k > 0$.*

□

Proposition 2.3.45. [DG70, Prop. IV.2.2.3]

(1) *Every k-subgroup and every quotient of a unipotent k-group is unipotent.*

(2) *All limits of unipotent k-groups are unipotent.*

□

Corollary 2.3.46. *The category of unipotent k-groups is equivalent to the pro-completion of the category of algebraic, unipotent k-groups.*

Proof. The proof is completely analogous the proof of Corollary 2.3.35. □

Corollary 2.3.47. *The categories of commutative unipotent k-groups and of algebraic commutative unipotent k-groups both form Abelian subcategories of $\mathbf{PAAGS}_k$.* □

If $\operatorname{char} k = 0$, then the categories of commutative unipotent k-groups and of algebraic commutative unipotent k-groups have a particularly simple structure.

Proposition 2.3.48. [DG70, Prop. IV.2.4.2] *Assume* $\operatorname{char} k = 0$.

(1) *The functor*

$$\begin{array}{rcl} \mathbf{Vect}_k & \to & \mathrm{Grp}(\mathbf{Sch}_k) \\ V & \mapsto & \operatorname{Spec} S(V) \end{array}$$

(where $S(V)$ denotes the symmetric algebra of V) restricts to an equivalence between the category of finite dimensional vector spaces over k and the category of algebraic commutative unipotent k-groups.

(2) *The functor*

$$\begin{array}{rcl} \mathrm{Grp}(\mathbf{Sch}_k) & \to & \mathbf{Vect}_k \\ G & \mapsto & \mathrm{Grp}(\mathbf{Sch}_k)(G, \mathbb{G}_a) \end{array}$$

(see Example 2.1.19) restricts to an anti-equivalence between the category of commutative unipotent k-groups and the category of vector spaces over k; it restricts further to an anti-equivalence between the category of commutative algebraic unipotent k-groups and the category of finite dimensional vector spaces over k.

□

In characteristic > 0 the theory is considerably richer; if k is also perfect, then the category of commutative unipotent k-groups is anti-equivalent to the category of so-called effaceable Dieudonné modules (see [DG70, Th. V.1.4.3]); this is a sub-category of the category of modules over the so-called Dieudonné ring over k (see [DG70, §V.1.3]). This is the main reason why we will work over a field of characteristic 0 in Chapters 4 & 5.

2.3.6 Decomposition of commutative affine group schemes

Theorem 2.3.49. [DG70, IV.3. Th. 1.1] *Let G be a commutative affine k-group, then G possesses a largest affine subgroup of multiplicative type G_m; this subgroup is normal, and the quotient $G_u := G/G_m$ is unipotent. If k is perfect, then the resulting short exact sequence*

$$0 \to G_m \to G \to G_u \to 0$$

splits. □

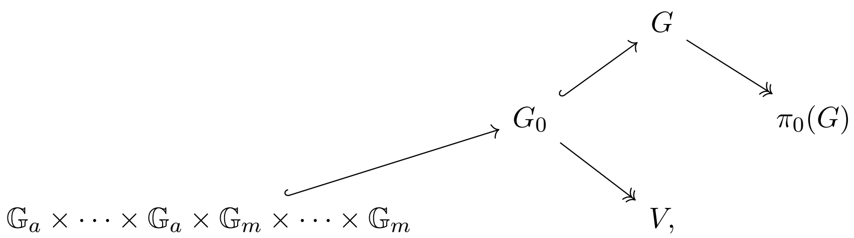
Notation 2.3.50. Let G be a commutative affine k-group, then its biggest multiplicative subgroup is denoted by G_m, and the unipotent quotient of G by G_m is denoted by G_u. ⌟

Proposition 2.3.51. [DG70, Prop. IV.3.1.3] *Let G, H be commutative affine k-groups and let $\varphi : G \to H$ be a morphism, then $\varphi(G_m) \subseteq H_m$, and if k is perfect, then $\varphi(G_u) \subseteq H_u$.* □

Proposition 2.3.52. *If k is algebraically closed and of characteristic zero, then any connected commutative algebraic affine k-group is of the form $\mathbb{G}_m \times \cdots \times \mathbb{G}_m \times \mathbb{G}_a \times \cdots \times \mathbb{G}_a$.*

Proof. This follows from putting together Theorem 2.3.24, Propositions 2.3.36 & 2.3.48.1 and Theorem 2.3.49. □

Summarising, if k is algebraically closed and of characteristic zero, then by Theorems 2.3.6, 2.3.24 and Propositions 2.3.52 for any commutative algebraic k-group G we obtain the following composition series



(2.1)

where V is an Abelian k-variety.

2.4 Sheaves associated to commutative group schemes

In this section, let S be a scheme and let X, T be two schemes over S, then the presheaf $\mathbf{Sch}_S(\_, T) : \mathbf{Sch}_{S,\mathrm{Zar}} \to \mathbf{Set}$ is a sheaf because Zar is subcanonical. The big site associated to $X \to S$ is isomorphic to $\mathbf{Sch}_{X,\mathrm{Zar}}$ (see [Stacks, Tag 03EH]), so that the sheaf $\mathbf{Sch}_S(\_, T)$ restricts to a sheaf on $\mathbf{Sch}_{X,\mathrm{Zar}}$ and then restricts further to a sheaf on $\mathbf{Ouv}_X$.

Definition 2.4.1. The sheaf $\mathbf{Sch}_S(\_, T) : \mathbf{Ouv}_X^{\mathrm{op}} \to \mathbf{Set}$ is called the *sheaf (of sets) on X associated to T*, and is denoted by T_X. ⌟

For any two S-schemes T', T'' and any morphism $\varphi : T' \to T''$ we obtain a morphism of sheaves on $\mathbf{Sch}_{S,\mathrm{Zar}}$ given by

$$\left\{ \begin{array}{rcl} \mathbf{Sch}_S(U,T') & \to & \mathbf{Sch}_S(U,T'') \\ f & \mapsto & \varphi \circ f \end{array} \right\}_{U \in \mathbf{Sch}_S},$$

which restricts to a morphism of sheaves on X so we obtain a morphism $T'_X \to T''_X$.
In the next subsection it will be important to consider fppf-sheaves. We obtain the commutative diagram of functors

$$\begin{array}{ccccc} \mathbf{Sch}_S & \longrightarrow & \widetilde{\mathbf{Sch}}_{S,\mathrm{fppf}} & \longrightarrow & \widetilde{\mathbf{Sch}}_{S,\mathrm{Zar}} & & \\ & & \downarrow & & \downarrow & & \\ & & \widetilde{\mathbf{Sch}}_{X,\mathrm{fppf}} & \longrightarrow & \widetilde{\mathbf{Sch}}_{X,\mathrm{Zar}} & \longrightarrow & \mathbf{Sh}_X. \end{array}$$

As all these functors are right exact, we obtain the commutative diagram

$$\begin{array}{ccccccc} \mathrm{Ab}(\mathbf{Sch}_S) & \longrightarrow & \mathrm{Ab}(\widetilde{\mathbf{Sch}}_{S,\mathrm{fppf}}) & \longrightarrow & \mathrm{Ab}(\widetilde{\mathbf{Sch}}_{S,\mathrm{Zar}}) & & \\ & & \downarrow & & \downarrow & & \\ & & \mathrm{Ab}(\widetilde{\mathbf{Sch}}_{X,\mathrm{fppf}}) & \longrightarrow & \mathrm{Ab}(\widetilde{\mathbf{Sch}}_{X,\mathrm{Zar}}) & \longrightarrow & \mathrm{Ab}(\mathbf{Sh}_X). \end{array} \tag{2.2}$$

Some properties of sheaves associated to special group schemes

Proposition 2.4.2. *Let k be a field, then sheaves on irreducible k-schemes associated to commutative constant k-groups are flasque.*

Proof. Let X be an irreducible k-scheme and let G be a commutative constant k-group corresponding to the Abelian group Γ, then G_X is given by $U \mapsto \mathbf{Top}(U, \Gamma)$ (see Example 2.1.21). As every open subset of X is connected, G_X is isomorphic to $U \mapsto \Gamma$ for $U \neq \varnothing$ and $\varnothing \mapsto 0$, which is clearly a flasque sheaf. □

Proposition 2.4.3. *Let k be a field, then sheaves on regular k-schemes associated to Abelian k-varieties are flasque.*

Proof. This is an immediate corollary of 2.3.22. □

2.4.1 Exactness of $\mathrm{Ab}(\mathbf{Sch}_k) \to \mathrm{Ab}(\widetilde{\mathbf{Sch}}_{k,\mathrm{Zar}})$

We now assume that S is the spectrum of a field k, and we will only consider Abelian (pre-) sheaves, so that $\widetilde{\mathcal{C}}_J$ denotes the Abelian category of Abelian sheaves on any site $(\mathcal{C}, J)$. All functors in (2.2) are left exact and the vertical functors are even exact (see [Stacks, Tag 00XZ]). From the description of how to construct quotient k-groups in §2.1.2 it is clear that the restriction of the functor $\mathrm{Ab}(\mathbf{Sch}_k) \to \widetilde{\mathbf{Sch}}_{k,\mathrm{fppf}}$ to $\mathbf{AGS}_k$ or $\mathbf{PAAGS}_k$ is exact and thus also its composition with $\widetilde{\mathbf{Sch}}_{k,\mathrm{fppf}} \to \widetilde{\mathbf{Sch}}_{X,\mathrm{fppf}}$. Unfortunately the restriction of the functor $\mathrm{Ab}(\mathbf{Sch}_k) \to \mathbf{Sh}_X$ to $\mathbf{AGS}_k$ or $\mathbf{PAAGS}_k$ is no longer exact. Keeping in mind that a complex $0 \to F' \to F \to F'' \to 0$

of Abelian Zariski sheaves is exact iff $0 \to F'(A) \to F(A) \to F''(A) \to 0$ is exact for every local k-algebra A, we consider the following example.

Example 2.4.4. Recall from Example 2.1.26 that we have a short exact sequence $0 \to \mu_{n,k} \to \mathbb{G}_{m,k} \to \mathbb{G}_{m,k} \to 0$. This sequence does not induce an exact sequence of Zariski sheaves, for if it did, then from any local k-algebra A we would obtain a short exact sequence $1 \to \mu_n(A) \to \mathbb{G}_m(A) \to \mathbb{G}_m(A) \to 1$, which is not true in general; take for example $A = k[X]_{(X)}$. ⌟

Fortunately the short exact sequences of commutative k-groups, which are most important to us, do induce short exact sequences of Zariski sheaves.

Proposition 2.4.5. *Assume that k is algebraically closed and let G be a locally algebraic k-group, then the short exact sequence*

$$0 \to G^\circ \to G \to \pi_0(G) \to 0$$

(see Theorem 2.3.6) induces an exact sequence of Zariski sheaves.

Proof. Because k is algebraically closed $\pi_0(G)$ is a constant k-group (see Corollary 2.3.3) and every closed point of G is rational. Every connected component of G contains a closed point (see [GW10, Rem. 3.35]). Let $x \in \pi_0(G)$ and let G_x be the corresponding connected component. We obtain the pull-back diagram

$$\begin{array}{ccc} G & \twoheadrightarrow & \pi_0(G) \\ \uparrow & & \uparrow x \\ G_x & \longrightarrow & \operatorname{Spec} k \end{array}$$

together with a section of $G_x \to \operatorname{Spec} k$, so we see that any morphism to x factors through $G \to \pi_0(G)$. □

Proposition 2.4.6. *Assume that k is algebraically closed of characteristic 0, then for any connected algebraic k-group G the unique short exact sequence*

$$0 \to H \to G \to A \to 0 \tag{2.3}$$

such that H is a connected affine algebraic k-group and A is an Abelian k-variety (see Theorem 2.3.24) induces an exact sequence of Zariski sheaves.

Proof. Let X be a k-scheme, then by the previous discussion we know that (2.3) induces a short exact sequence in $\widetilde{\mathbf{Sch}}_{X,\mathrm{fppf}}$, so we obtain a long exact sequence

$$\begin{array}{l} \hookrightarrow H^1_{\mathrm{fppf}}(X, H) \longrightarrow \cdots \\ 0 \longrightarrow \mathbf{Sch}_k(X, H) \longrightarrow \mathbf{Sch}_k(X, G) \longrightarrow \mathbf{Sch}_k(X, A) \end{array} .$$

We will show that $H^1_{\mathrm{fppf}}(X, H) \cong H^1_{\mathrm{Zar}}(X, H)$, and then we see that (2.3) is exact as the first Zariski cohomology group of any sheaf of Abelian groups vanishes on local k-schemes (see Corollary 3.1.13). By Proposition 2.3.52 the k-scheme H is isomorphic to a finite product $\mathbb{G}_a \times \cdots \times \mathbb{G}_a \times \mathbb{G}_m \times \cdots \times \mathbb{G}_m$, so it is enough to show $H^1_{\mathrm{fppf}}(X, \mathbb{G}_a) \cong H^1_{\mathrm{Zar}}(X, \mathbb{G}_a)$ and $H^1_{\mathrm{fppf}}(X, \mathbb{G}_m) \cong H^1_{\mathrm{Zar}}(X, \mathbb{G}_m)$. To see the first isomorphism we recall that for any quasi-coherent $\mathscr{O}_X$-module F and any $i \in \mathbb{N}$ we have $H^1_{\mathrm{fppf}}(X, F) \cong H^1_{\mathrm{Zar}}(X, F)$, where the first cohomology group is obtained from the fppf-sheaf $(i : U \to X) \mapsto \Gamma(U, i^*(F))$ (see [Mil80, Prop. III.3.7]); it is then easily verified that for any morphism $i : U \to X$ we have $\Gamma(U, i^*\mathscr{O}_X) \cong \Gamma(U, \mathscr{O}_U) \cong \mathbf{Sch}_k(U, \mathbb{G}_a)$. The second isomorphism is Hilbert's Thoerem 90 (see [Mil80, Prop. III.4.9]). □

Proposition 2.4.7. *Any split exact sequence of commutative k-groups induces an exact sequence of Zariski sheaves.*

Proof. This follows from the fact that all functors in (2.2) are additive. □

Chapter 3

Local Cohomology

3.1 Fundamental notions

In this section we recall some basic facts about sheaves and sheaf cohomology, and establish some conventions.

3.1.1 Inverse image functors, separable presheaves and flasque presheaves

Throughout this subsection X and Y denote topological spaces and $\varphi : X \to Y$ denotes a continuous map. We then obtain a functor $\mathbf{Ouv}_Y \to \mathbf{Ouv}_X$ given by $U \mapsto \varphi^{-1}(U)$, which induces the direct image functor $\varphi_* : \mathbf{PSh}_X \to \mathbf{PSh}_Y$. As **Set** admits all small colimits we see that the functor φ_* possesses a left adjoint (see [KS06, §2.3]), which we denote by $\varphi^\dagger$, and call the inverse image functor (of presheaves) of φ. As $\varphi^\dagger F$ is just the left Kan extension of F we see that it is given by $U \mapsto \varinjlim_{\varphi(U)\subseteq V} F(U)$.

Proposition 3.1.1. *The functor*

$$\varphi_* : \mathbf{PSh}_X \to \mathbf{PSh}_Y$$

as well as its left adjoint

$$\mathbf{PSh}_X \leftarrow \mathbf{PSh}_Y : \varphi^\dagger$$

take separated presheaves to separated presheaves, so that $\varphi^\dagger \dashv \varphi_*$ restricts to a new adjunction

$$\varphi_* : \mathbf{PSh}^s_X \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{PSh}^s_Y : \varphi^\dagger.$$

Proof. It is well known that φ_* takes separated presheaves to separated presheaves. To see that this is the case for $\varphi^\dagger$, let $U \subseteq X$ be an open subset and let $\{U_i\}_{i\in I}$ be an open cover of U; we must show that for any two sections $s, t \in \varphi^\dagger(U)$ such that for all $i \in I : s|_{U_i} = t|_{U_i}$ we have $s = t$. Assume that s and t are represented by the pairs $(\tilde{s}, V)$ and $(\tilde{t}, W)$ respectively; by restricting to $V \cap W$ we may assume that $V = W$; assume also that for every $i \in I$ the section $s|_{U_i} = t|_{U_i}$ is represented by (s_i, V_i), with $V_i \subseteq V$. By assumption there exists for every $i \in I$ an open subset $\varphi(U_i) \subseteq W_i \subseteq V_i$ such that $\tilde{s}|_{W_i} = s_i|_{W_i}$. The pairs $(\tilde{s}|_{\bigcup W_i}, \bigcup_{i\in I} W_i)$ and

$(\tilde{t}|_{\bigcup W_i}, \bigcup_{i\in I} W_i)$ still represent s and t respectively, but because $(\tilde{s}|_{\bigcup W_i})|_{W_i} = (\tilde{t}|_{\bigcup W_i})|_{W_i}$ for all $i \in I$ we see that $\tilde{s}|_{\bigcup W_i} = \tilde{t}|_{\bigcup W_i}$, so that $s = t$. □

Proposition 3.1.2. *The functor*

$$\varphi_* : \mathbf{PSh}_X \to \mathbf{PSh}_Y$$

as well as its left adjoint

$$\mathbf{PSh}_X \leftarrow \mathbf{PSh}_Y : \varphi^\dagger$$

take flasque presheaves to flasque presheaves, so that $\varphi^\dagger \dashv \varphi_*$ restricts to a new adjunction

$$\varphi_* : \mathbf{PSh}^f_X \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{PSh}^f_Y : \varphi^\dagger.$$

Proof. That φ_* takes flasque presheaves to flasque presheaves is obvious. To see that this is the case for $\varphi^\dagger$, let $U \subseteq X$ be an open subset, then any section $s \in \varphi^\dagger F(U)$ is represented by a pair $(\tilde{s}, Y)$; but $(\tilde{s}, Y)$ also represents a section in $\varphi^\dagger F(U')$ for any other open subset $U' \subseteq X$, in particular any such open subset containing U. □

3.1.2 Dimension and codimension

Dimension and codimension of topological spaces

Definition 3.1.3. Let X be a topological space, then the supremum of the lengths of the chains of irreducible closed subsets of X is called the *dimension of* X and is denoted by $\dim X$. ⌟

Remark 3.1.4. We immediately see that $\dim \varnothing = -\infty$. ⌟

Definition 3.1.5. Let X be a topological space and let $Y \subseteq X$ be an irreducible closed subset The supremum of the lengths of chains of irreducible closed subsets of X of which Y is the smallest element is called the *codimension of* Y and is denoted by $\operatorname{codim} Y$. If Y is any closed subset, then the infimum of the codimensions of the irreducible components of Y is called the *codimension of* Y and is denoted by $\operatorname{codim} Y$. ⌟

Remark 3.1.6. We immediately see that $\operatorname{codim} \varnothing = \infty$ (viewing $\varnothing$ as a subspace of X). ⌟

Notation 3.1.7. Let X be a topological space, then for all $\ell \geq 0$ the set of points such that their closure has codimension ℓ is denoted X^ℓ. ⌟

Dimension and codimension of schemes

Definition 3.1.8. Let A be a ring, then the *dimension of* A, denoted by $\dim A$, is the dimension of the underlying topological space of $\operatorname{Spec} A$. ⌟

Proposition 3.1.9. *Let X be a scheme and let Z be a closed irreducible subset with generic point z, then* $\operatorname{codim} Z = \dim \mathscr{O}_{X,z}$.

Proof. First we assume that X is the affine spectrum of a ring A, in which case the statement of the proposition is easy; if $z = \mathfrak{p} \in \operatorname{Spec} A$, then $Z = V(\mathfrak{p})$ and we have

$$\operatorname{codim} Z = \sup \left\{ \text{ length}(\mathfrak{p}_0 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \subsetneq \mathfrak{p}_n) \;\middle|\; \mathfrak{p}_i \in \operatorname{Spec} A,\ \mathfrak{p}_n = \mathfrak{p} \right\} = \dim A_{\mathfrak{p}}.$$

In general we have

$$\operatorname{codim} Z = \sup \left\{ \text{ length}(x_0, \dots, x_n) \;\middle|\; x_i \in X,\ x_0 = z,\ x_i \in \overline{\{x_{i+1}\}},\ x_i \neq x_{i+1} \right\}.$$

Let $x_0, \dots, x_n \in X$ such that $x_0 = z$, $x_i \in \overline{\{x_{i+1}\}}$, $x_i \neq x_{i+1}$, then every open neighbourhood and in particular every affine open neighbourhood of $x_0 = z$ contains all the elements x_i, so that the general case reduces to the affine case. □

Definition 3.1.10. Let A be a ring and $\mathfrak{a}$ an ideal of A, then the *height* of $\mathfrak{a}$, denoted $\operatorname{ht}(\mathfrak{a})$ is the number $\operatorname{codim}(V(\mathfrak{a}), \operatorname{Spec} A)$. ⌟

3.1.3 Sheaf cohomology

Sheaf cohomology of Abelian sheaves vs. sheaf cohomology of $\mathscr{O}_X$-modules

Let $(X, \mathscr{O}_X)$ be a ringed space. In standard references of local cohomology such as [SGA 2] the cohomology of the functors $\Gamma : \mathbf{Mod}_{\mathscr{O}_X} \to \mathbf{Ab}$ and $\Gamma : \mathbf{Sh}_X = \mathbf{Mod}_{\mathbb{Z}_X} \to \mathbf{Ab}$ are not distinguished. This is justified as by [God58, §II.7.1] the category of modules over a sheaf of rings contains enough injective sheaves and these are flasque; the latter notion does not depend on any module structure which may be present on an Abelian sheaf.

Sheaf cohomology on Noetherian spaces

Theorem 3.1.11 (Grothendieck)**.** [Gro57, Th. 3.6.5.] *Let X be a Noetherian topological space of dimension n, then $H^i(X, F) \cong 0$ for all $i > n$ and for all abelian sheaves F on X.* □

Sheaf cohomology on local schemes

Lemma 3.1.12. *Let $(A, \mathfrak{m})$ be a local ring, and G a sheaf of groups on X, then $\check{H}^1(\operatorname{Spec} A, G) \cong 0$.*

Proof. The set $\check{H}^1(\operatorname{Spec} A, G)$ is naturally bijective to the set of isomorphism classes of G-torsors on $\operatorname{Spec} A$. Let T be a G-torsor on $\operatorname{Spec} A$, then there exists a covering $\{U_i\}_{i \in I}$ of $\operatorname{Spec} A$ such that $T(U_i) \neq \varnothing$ for every $i \in I$, but the only open set containing $\mathfrak{m}$ is $\operatorname{Spec} A$, and a G-torsor is trivial iff $T(\operatorname{Spec} A) \neq 0$. □

Corollary 3.1.13. *Let $(A, \mathfrak{m})$ be a local ring, and F and a sheaf of Abelian groups on X, then $H^1(\operatorname{Spec} A, G) \cong 0$.*

Proof. As G is Abelian the pointed set $\check{H}^1(\operatorname{Spec} A, G)$ comes naturally endowed with the structure of an Abelian group which coincides with the first Zariksi cohomology group of G. □

3.2 Local cohomology

3.2.1 Fundamental notions

Definition 3.2.1. Let X be a topological space, let $Z \subseteq X$ be a closed subset, and let F be an Abelian sheaf on X, then $\Gamma_Z(X, F)$ denotes the subgroup of $\Gamma(X, F)$ consisting of those sections s with support in Z, i.e. such that for all $x \in X \setminus Z$ we have $s_x = 0$ or equivalently such that $s|_{X \setminus Z} = 0$. ⌟

Example 3.2.2. We consider two trivial cases. Let X be a topological space, let $Z \subseteq X$ be a closed subset, and let F be an Abelian sheaf on X. If $Z = X$ then $\Gamma_Z(X, F) = \Gamma(X, F)$ and if $Z = \varnothing$, then $\Gamma_Z(X, F) = 0$. ⌟

Example 3.2.3. Let k be a field, let X be an irreducible k-scheme, and let Z be a closed subset of X, then

$$\Gamma_Z(X, G_X) \cong \begin{cases} \Gamma(X, G_X) & \text{if} \quad Z = X, \\ 0 & \text{if} \quad Z \subsetneq X \end{cases}$$

for every commutative k-group scheme G. In particular, if X is a local k-scheme with closed point x, then

$$\Gamma_{\{x\}}(X, G_X) \cong \begin{cases} \Gamma(X, G_X) & \text{if} \quad \dim X = 0, \\ 0 & \text{if} \quad \dim X > 0. \end{cases}$$

To see this, let $s \in \Gamma(X, G_X)$, then $s \in \Gamma_Z(X, G_X)$ iff $s_{X \setminus Z} = 0$. The open subset $s^{-1}(G \setminus \{e\})$ (see Proposition 2.1.2) then intersects with $X \setminus Z$ iff $Z \neq X$ and $s \neq 0$. ⌟

Definition 3.2.4. Let X be a topological space and let $Z \subseteq X$ be a closed subset, then for every $i \geq 0$ the i-th derived functor of

$$\Gamma_Z(X, \_) : \mathbf{Sh}_X \to \mathbf{Ab}$$

is denoted by $H^i_Z(X, \_)$, and for any Abelian sheaf F on X the group $H^i_Z(X, F)$ is called the *i-th cohomology group of F with supports in Z.* ⌟

As in previous chapters we will give a more general account of the theory than is strictly necessary so that we may get a better feel of the subject. If X is locally path connected, then for any discrete topological Abelian group G the cohomology groups of the sheaf $X \supseteq U : U \mapsto \mathbf{Top}(U, G)$ are canonically isomorphic to the Eilenberg-Steenrod cohomology groups of X with coefficients in G (see [Voi02, Th. 4.47]). This result is easily extended to the relative case, i.e. for every $i \geq 0$ we have a canonical isomorphism $H^i_Z(X, G) \cong H^i(X, X \setminus Z; G)$. We will now show that local cohomology groups of arbitrary Abelian sheaves on arbitrary topological spaces satisfy many of the properties of Eilenberg-Steenrod cohomology groups.

Remark 3.2.5. Cohomology with supports in a closed subset may be generalised in various ways, e.g. to cohomology with supports in a locally closed subset or cohomology with supports in a family of closed subsets. For a systematic treatment on the variations on the theme of cohomology with supports see [Har66, Ch. IV]. ⌟

Proposition 3.2.6. *Let X be a topological space and let $Z \subseteq X$ be a closed subset, then for any short exact sequence of Abelian sheaves $0 \to F' \to F \to F'' \to 0$ on X we obtain a long exact sequence*

$$0 \longrightarrow H^0_Z(X, F') \longrightarrow H^0_Z(X, F) \longrightarrow H^0_Z(X, F'') \longrightarrow H^1_Z(X, F') \longrightarrow \cdots .$$

Proof. This is a fundamental property of left derived functors. □

Proposition 3.2.7 (Excision)**.** *Let X be a topological space, let $Z \subseteq X$ be a closed subset, and let F be an Abelian sheaf on X, then for any open subset $U \subseteq X$ containing Z and for all $i \geq 0$ the canonical homomorphism*

$$H^i_Z(X, F) \to H^i_Z(U, F|_U)$$

is an isomorphism.

Proof. We obtain a commutative diagram

$$\begin{array}{ccc} \Gamma(X \setminus Z, F) & \longrightarrow & \Gamma(U \setminus Z, F) \\ \uparrow & & \uparrow \\ \Gamma(X, F) & \longrightarrow & \Gamma(U, F) \\ \uparrow & & \uparrow \\ \Gamma_Z(X, F) & \longrightarrow & \Gamma_Z(U, F) \end{array}$$

in which we must show that the homomorphism $\Gamma_Z(X, F) \to \Gamma_Z(U, F)$ is an isomorphism. Let $s \in \Gamma_Z(U, F)$ and consider $0_{X\setminus Z} \in \Gamma(X \setminus Z, F)$, then $s|_{U\setminus Z} = 0_{X\setminus Z}|_{U\setminus Z} = 0|_{U\setminus Z}$, so there exists a unique element $t \in \Gamma(X, F)$ such that $t|_U = s$ and $t|_{X\setminus Z} = 0|_{U\setminus Z}$ by the sheaf conditions. □

Proposition 3.2.8 (Exactness)**.** *Let X be a topological space, let $Z \subseteq X$ be a closed subset, and let F be an Abelian sheaf on X, then there is a long exact seqeunce*

$$0 \longrightarrow \Gamma_Z(X, F) \longrightarrow \Gamma(X, F) \longrightarrow \Gamma(X \setminus Z, F) \longrightarrow H^1_Z(X, F) \longrightarrow \cdots,$$

which is functorial in F, and where the homomorphisms

$$0 \to \Gamma_Z(X, F) \to \Gamma(X, F) \to \Gamma(X \setminus Z, F)$$

are the canonical ones.

Proof. Let I be a flasque sheaf on X, then the sequence

$$0 \to \Gamma_Z(X, I) \to \Gamma(X, I) \to \Gamma(X \setminus Z, I) \to 0$$

is exact, so we see that for any injective resolution I^* of F we obtain an exact sequence of cochain complexes of Abelian groups (recall that injective sheaves are flasque; see [God58, §II.7.1])

$$\begin{array}{ccccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \Gamma_Z(X, I^1) & \longrightarrow & \Gamma(X, I^1) & \longrightarrow & \Gamma(X \setminus Z, I^1) & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \Gamma_Z(X, I^0) & \longrightarrow & \Gamma(X, I^0) & \longrightarrow & \Gamma(X \setminus Z, I^0) & \longrightarrow & 0; \end{array}$$

taking cohomology of this sequence yields the desired long exact sequence. □

Finally we show that we may compute local cohomology groups using flasque sheaves (see [KS06, Cor. 13.3.8]).

Proposition 3.2.9. [SGA 2, Cor. 2.12] *Let X be a topological space, let $Z \subseteq X$ be a closed subset, then for any flasque sheaf F on X we have*

$$H^i_Z(X, F) \cong 0 \qquad (i \geq 0).$$

□

Comparing local cohomology on different spaces

Given a continuous map $\varphi : X_1 \to X_2$ and a sheaf F on X_2, there is a well known canonical homomorphism $H^i(X_2, F) \to H^i(X_1, \varphi^* F)$ for all $i \in \mathbb{N}$ (see e.g. [Ive86, §II.5] or [Sch11a, (6.8)]); we will show how to extend this canonical homomorphism to local cohomology groups. Let $Z_1 \subseteq X_1$, $Z_2 \subseteq X_2$ be closed subspaces such that $\varphi^{-1}(Z_2) \subseteq Z_1$, and let I^* be an injective resolution of F and J^* an injective resolution of $\varphi^* F$, then there is a morphism cochain complexes $\varphi^* I^* \to J^*$ unique up to homotopy such that

$$\begin{array}{ccc} J^0 & \longleftarrow & \varphi^* I^0 \\ \uparrow & & \uparrow \\ \varphi^* F & \xleftarrow{=} & \varphi^* F \end{array}$$

commutes, and thus for each $i \in \mathbb{N}$ we obtain the diagram

$$\begin{array}{ccccc}
\Gamma(X_1 \setminus Z_1, J^i) & \longleftarrow & \Gamma(X_1 \setminus Z_1, \varphi^* I^i) & \longleftarrow & \Gamma(X_2 \setminus Z_2, I^i) \\
\uparrow & & \uparrow & & \uparrow \\
\Gamma(X_1, J^i) & \longleftarrow & \Gamma(X_1, \varphi^* I^i) & \longleftarrow & \Gamma(X_2, I^i) \\
\uparrow & & \uparrow & & \uparrow \\
\Gamma_{Z_1}(X_1, J^i) & \longleftarrow & \Gamma_{Z_1}(X_1, \varphi^* I^i) & \longleftarrow & \Gamma_{Z_1}(I^i, X_2),
\end{array}$$

which commutes by the universal property of kernels; by taking cohomology of cochain complexes we obtain for every $i \in \mathbb{N}$ a canonical homomorphism

$$H^i_{Z_2}(X_2, F) \to H^i_{Z_1}(X_1, \varphi^* F). \tag{3.1}$$

Let X be a topological space, let $Z_1, \ldots, Z_n$ be pairwise disjoint closed subsets, and write $Z := Z_1 \cup \cdots \cup Z_n$, then we have canonical morphisms $H^i_{Z_j}(X, F) \to H^i_Z(X, F)$ for all $i \geq 0$ and all $j \in \{1, \ldots, n\}$, which induce a homomorphism

$$\bigoplus_{j=1}^{n} H^i_{Z_j}(X, F) \to H^i_Z(X, F). \tag{3.2}$$

Proposition 3.2.10. *The homomorphism (3.2) is an isomorphism.*

Proof. As direct sums are exact in **Ab** we see that $F \mapsto \bigoplus H^i_{Z_j}(X, F)$ is the derived functor of $F \mapsto \bigoplus \Gamma_{Z_j}(X, F)$ so it is enough to show that (3.2) is an isomorphism for $i = 0$. In this case the map is given by $(s_1, \ldots, s_n) \mapsto s_1 + \cdots + s_n$. To see injectivity assume that $s_1 + \ldots + s_n = 0$, then for every $j \in \{1 + \ldots + n\}$ we have $(s_1, \ldots, s_n)|_{X \setminus (Z \setminus Z_j)} = s_j|_{X \setminus (Z \setminus Z_j)} = 0$ and $s_j|_{X \setminus Z_j} = 0$ by assumption so that $s_j = 0$ by the sheaf condition. To see surjectivity, let $s \in \Gamma_Z(X, F)$, then if for each $j \in \{1, \ldots, n\}$ we write s_j for the section obtained by glueing together $s|_{X \setminus Z \setminus Z_j}$ and $0|_{X \setminus Z_j}$, it is easily checked that $(s_1, \ldots, s_n)$ is sent to s. □

We now examine how local cohomology behaves under localisation. Recall that for a scheme X and a point $x \in X$ the underlying topological space of $\operatorname{Spec} \mathscr{O}_{X,x}$ is canonically homeomorphic to the subspace of X consisting of all points $x' \in X$ such that $x \in \overline{\{x'\}}$, which may equivalently be described as the intersection of all open sets containing x. Denote by $j : \operatorname{Spec} \mathscr{O}_{X,x} \hookrightarrow X$ the canonical inclusion; we want to understand how the local cohomology groups of any sheaf F on X relate to the local cohomology groups of its inverse image sheaf $j^* F$. As we will not require the full scheme structure we will assume that X is a Noetherian, sober topological space. We introduce the following notation.

Notation 3.2.11. Let $z \in X$ then we denote by X_z the subspace of X consisting of those points $x \in X$ such that $z \in \overline{\{x\}}$, or equivalently, the intersection all open sets containing z. ⌟

Lemma 3.2.12. *Any open subset $U \subset X$ is sober, and the map sending any irreducible closed subset $Z \subset U$ to $\overline{Z}$ is a bijection between irreducible closed subsets of U and irreducible closed* *subsets of X whose generic point is in U.*

Proof. The map taking points in U to their closure (in U) is obviously injective, because for any two points $x, y \in U$ such that $x \neq y$ there is a closed set $Y \subset X$ such that w.l.o.g. $x \in Y$, $y \neq Y$ and thus $x \in Y \cap U$, $y \neq Y \cap U$. To see surjectivity, let Z be an irreducible closed subset in U, then there exists a unique point z in X such that $\overline{Z} = \overline{\{z\}}$; any open set V such that $V \cap \overline{Z} \neq \varnothing$ must contain z because otherwise the closed set $X \setminus V$ would contain all of $\overline{Z}$, so we see that $z \in U$. The second part of the statement is now clear. □

Lemma 3.2.13. *For any point $z \in X$ the subspace X_z is sober.*

Proof. We see that the map taking points in X_z to their closure is injective by an analogous argument as in the previous lemma. To check surjectivity, let Z be an irreducible closed subset of X_z, then in the proof of the previous lemma we saw that any open subset intersecting $\overline{Z}$ must contain its generic point, and therefore X_z must contain its generic point. □

A key ingredient in the proof of Lemma 3.2.40 is the following lemma.

Lemma 3.2.14. *Let $z \in X$ and denote by $j : X_z \hookrightarrow X$ the canonical inclusion, then for any sheaf F on X, the presheaf $j^\dagger F$ is a sheaf.*

Proof. Because X is Noetherian it is enough to show for any open subset $U \subseteq X_z$ that $j^\dagger F$ satisfies the sheaf condition for a finite cover $\{U_i\}_{i=1}^n$ of U by open subsets. By Proposition 3.1.1 the presheaf $j^\dagger F$ is separated, so it is enough to show for any family of sections $(s_i) \in \prod_{i=1}^n j^\dagger F(U_i)$, such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in \{1, \ldots, n\}$ that there exists an element $s \in j^\dagger F(U)$ such that $s|_{U_i} = s_i$ for all $i \in \{1, \ldots, n\}$. So let $(s_i) \in \prod_{i=1}^n j^\dagger F(U_i)$ be a family of sections as above, then there exist open sets $V_1, \ldots, V_n \subseteq X$ and a family of sections $(t_i) \in \prod_{i=1}^n F(V_i)$ such that $U_i = X_z \cap V_i$ and $s_i = [(t_i, V_i)]$ for all $i \in \{1, \ldots, n\}$; furthermore, for each pair (i, j) in $\{1, \ldots, n\}$ there exists an open subset $W_{ij} \subseteq V_i \cap V_j$ such that $U_i \cap U_j = X_z \cap W_{ij}$ and $t_i|_{W_{ij}} = t_j|_{W_{ij}}$. Denote by $z_1, \ldots, z_m$ the generic points of the irreducible components of the sets $(V_i \cap V_j) \setminus W_{ij}$ for all pairs (i, j) in $\{1, \ldots, n\}$, and write $Z := \overline{\{z_1\}} \cup \cdots \cup \overline{\{z_m\}}$, where the closures are taken in X. We claim that $Z \cap X_z = \varnothing$; indeed, if this were not the case, then there would exist an element z_i such that $\overline{\{z_i\}} \cap X_z \neq \varnothing$ and we would have:

$$\begin{aligned} X_z \cap \overline{\{z_i\}} \neq \varnothing &\iff \forall\, U \ni z :\; U \cap \overline{\{z_i\}} \\ &\iff \forall\, U \ni z :\; z_i \in U \\ &\iff z_i \in X_z, \end{aligned}$$

but this would contradict the fact that $(V_i \cap V_j \setminus W_{ij}) \cap X_z = \varnothing$ for each pair (i, j) in $\{1, \ldots, n\}$. Now, $s_i = [(t_i|_{V_i \setminus Z}, V_i \setminus Z)]$ for each $i \in \{1, \ldots, n\}$, and $(t_i|_{V_i \setminus Z})|_{W_{ij} \setminus Z} = (t_j|_{V_j \setminus Z})|_{W_{ij} \setminus Z}$, so there exists an element $t \in F((V_1 \cup \cdots \cup V_n) \setminus Z)$ such that $t|_{V_i \setminus Z} = t_i|_{V_i \setminus Z}$ for each $i \in \{1, \ldots, n\}$; setting $s := [(t, (V_1 \cup \cdots \cup V_n) \setminus Z)]$ we have $s_i = s|_{U_i}$ for each $i \in \{1, \ldots, n\}$. □

By applying Proposition 3.1.2 we obtain the following corollary:

Corollary 3.2.15. *If F is a flasque sheaf, then so is $j^\dagger F$.* □

3.2.2 Local cohomology of local rings

In this subsection A denotes a ring, and M an A-module.

Dimension of modules

Definition 3.2.16. The *dimension* of M is the dimension of $A/\operatorname{Ann}(M)$, and is denoted by $\dim M$. ⌟

By [Bou61, Prop. II.4.17] we see that if M is finitely generated then

$$\dim M = \dim(\operatorname{supp} M) \leq \dim A.$$

Example 3.2.17. Viewing A as a module over itself, we see that $\dim A = \dim(\operatorname{supp} A) = \dim(\operatorname{Spec} A)$, so that the dimension of A coincides with its dimension in the sense of Definition 3.1.3. ⌟

Convention 3.2.18. In light of Example 3.2.17 we will simply speak of the dimension of a ring, that is, we will not distinguish between its dimension in the sense of Definition 3.1.3 and its dimension in the sense of Definition 3.2.16. ⌟

Depth

Definition 3.2.19. An *M-regular sequence* is a sequence of elements $x_1, \ldots, x_n \in A$ such that x_1 is not a zero-divisor for M and for each $i \in \{2, \ldots, n\}$ the element x_i is not a zero-divisor for $M/(x_1, \ldots, x_{i_1})$. ⌟

Definition 3.2.20. Suppose that A is Noetherian and local with maximal ideal $\mathfrak{m}$, and that M is finitely generated, then the *depth* of M is the length of the longest M-regular sequence contained in $\mathfrak{m}$. ⌟

Proposition 3.2.21. [EGA IV$_1$, Prop. 16.4.6.ii] *Suppose that A is Noetherian and local with maximal ideal $\mathfrak{m}$, and that M is finitely generated, then*

$$\operatorname{depth} M \leq \dim M.$$

□

Regular local ring

Proposition 3.2.22. [EGA IV$_1$, Prop. 17.1.1] *Suppose that A is Noetherian and local with maximal ideal $\mathfrak{m}$. Let n be the dimension of A, then the following are equivalent:*

(I) *The dimension of the $A/\mathfrak{m}$-vector space $\mathfrak{m}^2/\mathfrak{m}$ is n.*

(II) *The ideal $\mathfrak{m}$ may be generated by n elements.*

(III) *The ideal $\mathfrak{m}$ may be generated by a sequence of elements which form an A-regular sequence.*

Definition 3.2.23. Assume that A is local and Noetherian of dimension n, then we say that A is a *regular local ring* if it satisfies the equivalent properties of Proposition 3.2.22. ⌟

Corollary 3.2.24. *If A is a regular local ring, then* $\operatorname{depth} A = \dim A$. □

Definition 3.2.25. Suppose that A is Noetherian and that M is finitely generated, then M is called *Cohen-Macaulay* if for every prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ we have $\operatorname{depth} M_{\mathfrak{p}} = \dim M_{\mathfrak{p}}$. ⌟

Remark 3.2.26. Cohen-Macaulay modules are extensively studied in their own right; see e.g. [BH93]. ⌟

Example 3.2.27. Regular local rings of dimension 0 are exactly fields and regular local rings of dimension 1 exactly DVRs. ⌟

Theorem 3.2.28 (Auslander-Buchsbaum)**.** [Mat89, Th. 20.3] *Every regular local ring is a UFD.* □

Vanishing of local cohomology groups

Theorem 3.2.29 (Grothendieck)**.** [SGA 2, Ex. III.3.4 & Th. 5.3.1] *Assume that A is Noetherian and local with maximal ideal $\mathfrak{m}$, and that M is finitely generated, then*

$$\begin{array}{rcll} H^i_{\{\mathfrak{m}\}}(M) & \cong & 0 & \text{if } i \notin [\operatorname{depth} M, \dim M] \text{ and} \\ H^i_{\{\mathfrak{m}\}}(M) & \ncong & 0 & \text{if } i = \dim M \text{ or } i = \operatorname{depth} M. \end{array}$$

□

Corollary 3.2.30. *If M is Cohen-Macaulay, then $H^i_{\{\mathfrak{m}\}}(M)$ vanishes for all $i \geq 0$ except for $i = \dim M = \operatorname{depth} M$.* □

3.2.3 The Cousin resolution

As outlined in the introduction of this thesis, the first step in the construction of the pro-algebraic resolution is to construct a certain flasque resolution of every sheaf associated to every commutative algebraic k-group. Now, for any Abelian sheaf F on a locally Noetherian sober topological space X one may construct the so-called Cousin complex C^* of flasque sheaves on X together with an augmentation morphism $F \to C^*$ (viewing F as cochain complex concentrated in degree 0). The Cousin complex is characterised up to unique isomorphism by the following three properties:

(a) For each $i \geq 0$ and for each $x \in X^i$ there exists an Abelian group M_x such that we have $C^i \cong \bigoplus_{i \in X^i} (i_x)_*(M_x)$, where i_x denotes the canonical inclusion $\{x\} \hookrightarrow X$.

(b) For each $i \geq 0$ the support of $H^i(C^*)$ lies in $\bigcup_{j \geq i+2} X^i$.

(c) The support of the kernel of the morphism $F \to H^0(C^*)$ lies in $\bigcup_{j \geq 1} X^i$ and the support the cokernel lies in $\bigcup_{j \geq 2} X^i$.

(see [Har66, Prop. IV.2.3])[1].

The Cousin complex is named after Pierre Cousin who studied certain questions arising in several complex variables in [Cou95] which were later addressed using cohomological methods in the 1950s (see e.g. [GR65, §I.E & §VIII.A]); the complex defined in [Har66] is a generalisation of the algebro-geometric analogue of a certain flasque resolution arising in these cohomological investigations which we discuss below in Example 3.2.33. The Cousin resolution is sometimes called the "Gersten resolution" or "Gersten-Quillen resolution" (e.g. in [Blo10]) after Steven Gersten, who defined an analogous complex in algebraic K-theory in [Ger73], for which he conjectured a certain exactness result, and after Daniel Quillen who provided a partial verification of this conjecture in [Qui73].

Returning to the question of resolving Abelian sheaves, we see from the properties characterising the Cousin complex of F up to unique isomorphism that F possesses at most one resolution where for every $i \geq 0$ the i-th sheaf is of the form $\bigoplus_{x \in X^i} (i_x)_*(M_x)$ for suitable Abelian groups M_x. We check the uniqueness of the groups M_x ($x \in X$) by hand: If we plug this resolution into $\varinjlim_{x \in U \text{open}} \Gamma_{U \cap \overline{\{x\}}}(U, F|_U)$ for a given $x \in X$ to compute its derived functor $\varinjlim_{x \in U \text{open}} H^i_{U \cap \overline{\{x\}}}(U, F|_U)$ (filtered colimits of Abelian groups are exact, see [KS06, Th. 3.1.6]) we see that $\varinjlim_{x \in U \text{open}} H^i_{U \cap \overline{\{x\}}}(U, F|_U)$ is isomorphic to M_x if $x \in X^i$ and isomorphic to 0 otherwise, that is, such a sheaf F possesses the following property:

Definition 3.2.31. A sheaf F on a locally Noetherian, sober topological space X is called *Cohen-Macaulay* if for every $x \in X$:

$$\varinjlim_{x \in U \text{open}} H^i_{U \cap \overline{\{x\}}}(U, F|_U) \cong 0 \quad (i \neq \operatorname{codim} \overline{\{x\}}).$$

⌟

Remark 3.2.32. Let M be a finitely generated module over a Noetherian ring, then we see that $\tilde{M}$ is Cohen-Macaulay iff M is Cohen-Macaulay (see Definition 3.2.26). ⌟

It turns out the converse is also true; that is, the Cousin complex of a Cohen-Macaulay sheaf is a flasque resolution (see [Har66, Prop. IV.2.6]). The rest of this subsection deals with proving this converse statement using only fairly elementary methods (in particular avoiding spectral sequences). We will assume that X is Noetherian and we will construct a complex of sheaves $F \to F^0 \to F^1 \to \cdots$ on X such that F^i is flasque with supports in X^i for every $i \geq 0$; it is exact when F is Cohen-Macaulay, so in this case it coincides with the Cousin resolution. Our construction is based on [MVW06, Th. 24.11].

Before we begin we give the following motivating example.

Example 3.2.33 (Grothendieck)**.** [Gro57, §3.4] Let X be an irreducible, Noetherian, regular

[1] The descending chain of subsets $X = \bigcup_{\ell \geq 0} X^\ell \supseteq \bigcup_{\ell \geq 1} X^\ell \supseteq \cdots$ is called the *arithmetic filtration* of X (see e.g. [Gro69]) and the condition cited above may be formulated as $C^i_x \cong 0$ for every $x \notin \bigcup_{\ell \geq i} X^\ell \setminus \bigcup_{\ell \geq i+1} X^\ell$ and every $i \geq 0$. The complex discussed in [Har66, Prop. IV.2.3] is constructed for any filtration $X = Z^0 \supseteq Z^1 \supseteq \cdots$ such that for every $\ell \geq 0$ the set Z^ℓ is closed under specialisation and such that every point in $Z^\ell \setminus Z^{\ell+1}$ is maximal (w.r.t. to specialisation).

scheme, then the Cousin resolution of $\mathscr{O}_X^\times$ is given by the so-called *divisor sequence*

$$\mathscr{O}_X^\times \to K_X^\times \to Z_X^1, \tag{3.3}$$

which we now proceed to describe.

The presheaf K_X sends every non-empty open subset of X to $K(X)$, the field of rational functions on X, and the empty set to 0. To see that K_X is a flasque sheaf it is a enough to note that it is the skyscraper sheaf at the generic point of X associated to $K(X)$.

Next, for every open subset $U \subseteq X$ the group of Weil divisors on U may be defined[2] as

$$Z_X^1(U) := \bigoplus_{x \in U^1} \mathbb{Z}.$$

For every two open subsets $V \subseteq U \subseteq X$ there is a restriction homomorphism $Z_X^1(U) \to Z_X^1(V)$ given by $\sum_{x \in U^1} n_x \mapsto \sum_{x \in V^1} n_x$. It is straightforward to check that the maps $U \mapsto Z_X^1(U)$ together with the restriction morphisms just described form a presheaf, which we again denote by Z_X^1. Similarly as for K_X we see that Z_X^1 is a flasque sheaf by noting that it is canonically isomorphic to the coproduct of skyscraper sheaves $\coprod_{x \in X^1} \mathbb{Z}_x$.

We now describe the morphisms of the divisor sequence.

The map $\mathscr{O}_X^\times \to K_X^\times$ is simply the restriction of the canonical inclusion $\mathscr{O}_X \hookrightarrow K_X$.

To construct the morphism $K_X^\times \to Z_X^1$ we first recall the following fact: Let F and G be sheaves (of sets) on a space S, then to define a morphism $F \to G$, it is enough to specify maps $F(U) \to G(U)$ compatible with the respective restriction maps of F and G for some *basis* of the topology of S (see e.g. [EGA I, §**0**.3.2]). Now, let $U \subseteq X$ be an open affine subset of X, and A a ring such that $U \cong \operatorname{Spec} A$. Firstly we note that $Z_X^1(U) \cong \bigoplus_{\mathfrak{p} \in \operatorname{Spec} A,\, \operatorname{ht}(\mathfrak{p})=1} \mathbb{Z}$ (see Proposition 3.1.9), and secondly that for all $\mathfrak{p} \in \operatorname{Spec} A$ such that $\operatorname{ht}(\mathfrak{p}) = 1$ the ring $A_\mathfrak{p}$ is a regular local ring of dimension 1, that is, a DVR. Via the composition of the canonical homomorphisms $A \xrightarrow{\cong} \mathscr{O}_X(U) \hookrightarrow K(X)$ we may view A and all its localisations as subrings of $K(X) \cong \operatorname{Frac} A$; then to each $\mathfrak{p} \in \operatorname{Spec} A$ such that $\operatorname{ht}(\mathfrak{p}) = 1$ there corresponds a unique valuation $\operatorname{ord}_\mathfrak{p} : K(X)^\times \to \mathbb{Z}$, such that $A_\mathfrak{p} = \left\{ f \in K(X) \;\middle|\; \operatorname{ord}_\mathfrak{p}(f) \geq 0 \right\} \cup \{0\}$. Every Noetherian ring possesses a finite number of minimal prime ideals, and thus every element $f \in A \setminus \{0\}$ is contained in at most finitely many prime ideals of height one, since these are the minimal non-zero prime ideals of A containing f, and these correspond to the minimal prime ideals of $A/(f)$. We may now define a homomorphism $K(X)^\times \xrightarrow{\oplus \operatorname{ord}_\mathfrak{p}} \bigoplus_{\mathfrak{p} \in \operatorname{Spec} A,\, \operatorname{ht}(\mathfrak{p})=1} \mathbb{Z}$, and thus a morphism $K(X)^\times \to Z_X^1(U)$ for each open affine subset U of X. It is somewhat tedious but straightforward to show that this family of homomorphisms is compatible with the restriction homomorphisms of K_X and Z_X^1. For each prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ we have $A^\times \subseteq A_\mathfrak{p}^\times$ and thus we obtain a cochain complex as in (3.3).

We must now verify that this complex is exact.

It is exact at $\mathscr{O}_X^\times$, since $\mathscr{O}_X^\times \to K_X^\times$ is a monomorphism.

[2]This definition is equivalent to the usual definition of Weil divisors (when restricted to Noetherian schemes), as may be found e.g. in [EGA IV$_4$, §21.6.3].

To see that the complex (3.3) is exact at $K_X^\times$, let x be any point in X, $U \cong \operatorname{Spec} A$ an affine subset of X containing x, and $\mathfrak{p}_x$ the prime ideal in $\operatorname{Spec} A$ corresponding to x, then, localising at x, we obtain the sequence $A_{\mathfrak{p}_x}^\times \hookrightarrow K(X)^\times \xrightarrow{\oplus \operatorname{ord}_\mathfrak{p}} \bigoplus_{\mathfrak{p} \in \operatorname{Spec} A_{\mathfrak{p}_x},\, \operatorname{ht}(\mathfrak{p})=1} \mathbb{Z}$ from the sequence $A^\times \hookrightarrow K(X)^\times \xrightarrow{\oplus \operatorname{ord}_\mathfrak{p}} \bigoplus_{\mathfrak{p} \in \operatorname{Spec} A,\, \operatorname{ht}(\mathfrak{p})=1} \mathbb{Z}$. Now, any Noetherian integral domain B is integrally closed iff $B = \bigcap_{\mathfrak{p} \in \operatorname{Spec} B,\, \operatorname{ht}(\mathfrak{p})=1} B_\mathfrak{p}$ (viewing B and its localisations as subrings of $\operatorname{Frac} B$) (see [Rei95, Th. 8.10]), and by Theorem 3.2.28 every regular local ring is a UFD, and a fortiori an integrally closed integral domain. Thus for every element $f/g \in (\operatorname{Frac} A_{\mathfrak{p}_x})^\times \cong (\operatorname{Frac} A)^\times \cong K(X)^\times$ we have $\oplus \operatorname{ord}_\mathfrak{p} f/g = 0$ iff f/g is contained in $(A_{\mathfrak{p}_x})_\mathfrak{p}^\times \cong A_\mathfrak{p}^\times$ for all $\mathfrak{p} \in \operatorname{Spec} A_{\mathfrak{p}_x}$ of height 1 iff $f/g \in A_{\mathfrak{p}_x}^\times$.
To verify exactness at Z_X^1 let x, U, A and $\mathfrak{p}_x$ be as before; we must show that $K(X)^\times \to \bigoplus_{\mathfrak{p} \in \operatorname{Spec} A_{\mathfrak{p}_x},\, \operatorname{ht}(\mathfrak{p})=1} \mathbb{Z}$ is surjective. Now, any Noetherian integral domain B is a UFD iff all its prime ideals of height one are principal (see [Mat89, Th. 20.1]), and thus, if we assume that B is normal, then $(\operatorname{Frac} B)^\times \to \bigoplus_{\mathfrak{p} \in \operatorname{Spec} B,\, \operatorname{ht}(\mathfrak{p})=1} \mathbb{Z}$ is surjective iff B is a UFD. We have already mentioned, that by Theorem 3.2.28 $A_{\mathfrak{p}_x}$ is a UFD, so we are done. ⌟

Remark 3.2.34. The sheaf of Weil divisors may be defined for arbitrary locally Noetherian schemes, and if moreover the scheme is normal, one can construct a cochain complex like the one in Example 3.2.33 (see [EGA IV$_4$, §21.6.3]), which is again exact iff all the stalks of the structure sheaf of the scheme are UFDs (see [EGA IV$_4$, Th. 21.6.9]). ⌟

Until the end of the section X will denote a Noetherian, sober space and F a sheaf on X. We begin with a few preliminary constructions.

Preliminary constructions

Let $i, \ell \in \mathbb{N}$, then there is a canonical homomorphism

$$\varinjlim_{\substack{T,Z \text{ closed} \subseteq X:\\ \operatorname{codim} T > \ell,\\ \operatorname{codim} Z = \ell}} H^i_{Z \setminus T}(X \setminus T, F|_{X \setminus T}) \to \coprod_{x \in X^\ell} \varinjlim_{x \in U \text{open}} H^i_{U \cap \overline{\{x\}}}(U, F|_U) \tag{3.4}$$

natural in F, which we now construct.
First, we show that $(T, Z) \mapsto H^i_{Z \setminus T}(X \setminus T, F|_{X \setminus T})$ is a functor, so that it makes sense to speak of the colimit $\varinjlim H^i_{Z \setminus T}(X \setminus T, F|_{X \setminus T})$. To this end, let (T_1, Z_1), (T_2, Z_2) be pairs of closed subsets of X such that $\operatorname{codim} T_1$, $\operatorname{codim} T_2 > \ell$, $\operatorname{codim} Z_1 = \operatorname{codim} Z_2 = \ell$ and $T_1 \subseteq T_2$, $Z_1 \subseteq Z_2$. Setting

$i = 0$ we obtain the following diagram, which commutes by the universal property of kernels:

$$\begin{array}{ccccccccc}
\Gamma(X \setminus (T_1 \cup Z_1), F) & & & & \longrightarrow & & & & \Gamma(X \setminus (T_2 \cup Z_1), F) \\
\Big\downarrow & \nwarrow & & & & & & \nearrow & \Big\downarrow \\
 & & \Gamma(X \setminus T_1, F) & & \longrightarrow & & \Gamma(X \setminus T_2, F) & & \\
 & & \Big\downarrow & \nwarrow & & \nearrow & \Big\downarrow & & \\
 & & & \Gamma_{Z_1 \setminus T_1}(X \setminus T_1, F) & \longrightarrow & \Gamma_{Z_1 \setminus T_2}(X \setminus T_2, F) & & & \\
 & & & \Big\downarrow & & \Big\downarrow & & & \\
 & & & \Gamma_{Z_2 \setminus T_1}(X \setminus T_1, F) & \longrightarrow & \Gamma_{Z_2 \setminus T_2}(X \setminus T_2, F) & & & \\
 & & \Big\downarrow & \swarrow & & \searrow & \Big\downarrow & & \\
 & & \Gamma(X \setminus T_1, F) & & \longrightarrow & & \Gamma(X \setminus T_2, F) & & \\
\Big\downarrow & \swarrow & & & & & & \searrow & \Big\downarrow \\
\Gamma(X \setminus (T_1 \cup Z_2), F) & & & & \longrightarrow & & & & \Gamma(X \setminus (T_2 \cup Z_2), F).
\end{array}$$

This provides us with a homomorphism $\Gamma_{Z_1 \setminus T_1}(X \setminus T_1, F) \to \Gamma_{Z_2 \setminus T_2}(X \setminus T_2, F)$ (it is in fact the canonical morphism (3.1)). If we consider a further pair of closed subsets (T_3, Z_3) of X such that $\operatorname{codim} T_3 > \ell$, $\operatorname{codim} Z_3 = \ell$ and $T_2 \subseteq T_3$, $Z_3 \subseteq Z_2$, and then construct the analogous homomorphism $\Gamma_{Z_1 \setminus T_1}(X \setminus T_1, F) \to \Gamma_{Z_3 \setminus T_3}(X \setminus T_3, F)$, we see that this morphism may also be obtained from composing $\Gamma_{Z_1 \setminus T_1}(X \setminus T_1, F) \to \Gamma_{Z_2 \setminus T_2}(X \setminus T_2, F)$ and $\Gamma_{Z_2 \setminus T_2}(X \setminus T_2, F) \to \Gamma_{Z_3 \setminus T_3}(X \setminus T_3, F)$, as the following diagram commutes:

$$\begin{array}{ccccc}
\Gamma_{Z_1 \setminus T_1}(X \setminus T_1, F) & \longrightarrow & \Gamma_{Z_1 \setminus T_2}(X \setminus T_2, F) & \longrightarrow & \Gamma_{Z_1 \setminus T_3}(X \setminus T_3, F) \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
\Gamma_{Z_2 \setminus T_1}(X \setminus T_1, F) & \longrightarrow & \Gamma_{Z_2 \setminus T_2}(X \setminus T_2, F) & \longrightarrow & \Gamma_{Z_2 \setminus T_3}(X \setminus T_3, F) \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
\Gamma_{Z_3 \setminus T_1}(X \setminus T_1, F) & \longrightarrow & \Gamma_{Z_3 \setminus T_2}(X \setminus T_2, F) & \longrightarrow & \Gamma_{Z_3 \setminus T_3}(X \setminus T_3, F).
\end{array}$$

Replacing F by an injective resolution in the two diagrams above and taking cohomology of the resulting cochain complexes we conclude that $(T, Z) \mapsto H^i_{Z \setminus T}(X \setminus T, F|_{X \setminus T})$ is a functor for $i \geq 0$.

Now we show that for all $i, \ell \geq 0$ the group $\coprod_{x \in X^\ell} \varinjlim H^i_{U \cap \overline{\{x\}}}(U, F|_U)$ may be realised as the vertex of a cocone on the functor $(T, Z) \mapsto H^i_{Z \setminus T}(X \setminus T, F|_{X \setminus T})$, so that (3.4) is then given by the universal property of $\varinjlim H^i_{Z \setminus T}(X \setminus T, F|_{X \setminus T})$. We again reduce to the case $i = 0$. For every pair of closed subsets (T, Z) of X such that $\operatorname{codim} T > \ell$, $\operatorname{codim} Z = \ell$ and for every $x \in (Z \setminus T)^\ell$ there is a canonical homomorphism $\Gamma_{Z \setminus T}(X \setminus T, F) \to F_x$ given by composing $\Gamma_{Z \setminus T}(X \setminus T, F) \hookrightarrow \Gamma(X \setminus T, F) \to F_x$; we must show that this homomorphism factors through $\varinjlim \Gamma_{U \cap \overline{\{x\}}}(U, F) \hookrightarrow F_x$. Denote by $Z_1, \ldots, Z_n$ the irreducible components of $Z \setminus T$ other than $\overline{\{x\}} \setminus T$. There then exists an open neighbourhood U of x contained in $X \setminus T$ such that $U \cap Z_i = \varnothing$

for all $i \in \{1, \dots, n\}$ (e.g. $(X \setminus T) \setminus (Z_1 \cup \dots \cup Z_n)$). As the diagram

$$\begin{array}{ccc} \Gamma(X \setminus T, F) & \longrightarrow & \Gamma((X \setminus T) \setminus Z, F) \\ \big\downarrow & & \big\downarrow \\ \Gamma(U, F) & \longrightarrow & \Gamma(U \setminus \overline{\{x\}}, F) \end{array}$$

commutes, the homomorphism $\Gamma(X \setminus T, F) \to \Gamma(U, F)$ then restricts to a homomorphism $\Gamma_{Z \setminus T}(X \setminus T, F) \to \Gamma_{\overline{\{x\}} \cap U}(U, F)$. We thus obtain a homomorphism

$$\Gamma_{Z \setminus T}(X \setminus T, F) \to \prod_{x \in (Z \setminus T)^\ell} \varinjlim_{x \in U \text{open}} \Gamma_{U \cap \overline{\{x\}}}(U, F).$$

Now the set $(Z \setminus T)^\ell$ is finite (every $\overline{\{x\}} \setminus T$ with $x \in (Z \setminus T)^\ell$ is an irreducible component of $Z \cap (X \setminus T)$), so that

$$\coprod_{x \in (Z \setminus T)^\ell} \varinjlim_{x \in U \text{open}} \Gamma_{U \cap \overline{\{x\}}}(U, F) \hookrightarrow \prod_{x \in (Z \setminus T)^\ell} \varinjlim_{x \in U \text{open}} \Gamma_{U \cap \overline{\{x\}}}(U, F)$$

is an isomorphism; composing

$$\Gamma_{Z \setminus T}(X \setminus T, F) \to \coprod_{x \in (Z \setminus T)^\ell} \varinjlim_{x \in U \text{open}} \Gamma_{U \cap \overline{\{x\}}}(U, F) \hookrightarrow \coprod_{x \in X^\ell} \varinjlim_{x \in U \text{open}} \Gamma_{U \cap \overline{\{x\}}}(U, F)$$

yields a homomorphism

$$\Gamma_{Z \setminus T}(X \setminus T, F) \to \coprod_{x \in X^\ell} \varinjlim_{x \in U \text{open}} \Gamma_{U \cap \overline{\{x\}}}(U, F). \tag{3.5}$$

It now remains to show that for any two pairs (T_1, Z_1), (T_2, Z_2) such that $T_1 \subseteq T_2$ and $Z_1 \subseteq Z_2$ the triangle

$$\begin{array}{ccc} & \coprod_{x \in X^\ell} \varinjlim_{x \in U \text{open}} \Gamma_{U \cap \overline{\{x\}}}(U, F) & \\ \nearrow & & \nwarrow \\ \Gamma_{Z_1 \setminus T_1}(X \setminus T_1, F) & \longrightarrow & \Gamma_{Z_2 \setminus T_2}(X \setminus T_2, F) \end{array}$$

commutes. First we note that both arrows with target $\coprod_{x \in X^\ell} \varinjlim \Gamma_{U \cap \overline{\{x\}}}(U, F)$ factor through $\coprod_{x \in (Z_2 \setminus T_1)^\ell} \varinjlim \Gamma_{U \cap \overline{\{x\}}}(U, F)$, which is isomorphic to $\prod_{x \in (Z_2 \setminus T_1)^\ell} \varinjlim \Gamma_{U \cap \overline{\{x\}}}(U, F)$, so that it is enough to show that for every $x \in (Z_1 \setminus T_1)^\ell$ the triangle

$$\begin{array}{ccc} & \varinjlim_{x \in U \text{open}} \Gamma_{U \cap \overline{\{x\}}}(U, F) & \\ \nearrow & & \nwarrow \\ \Gamma_{Z_1 \setminus T_1}(X \setminus T_1, F) & \longrightarrow & \Gamma_{Z_2 \setminus T_2}(X \setminus T_2, F) \end{array}$$

commutes; but this is immediate as this triangle is just the restriction of the triangle

$$\begin{array}{ccc} & F_x & \\ \nearrow & & \nwarrow \\ \Gamma(X \setminus T_1, F) & \longrightarrow & \Gamma(X \setminus T_2, F). \end{array}$$

The posets $\left\{ T \subseteq X \;\middle|\; T \text{ closed, } \operatorname{codim} T > \ell \right\}$ and $\left\{ Z \subseteq X \;\middle|\; Z \text{ closed, } \operatorname{codim} T = \ell \right\}$ (with the ordering given by inclusion) are filtered, and their product is then also filtered. As both filtered colimits and coproducts are exact in **Ab** (see [KS06, Cor. 3.1.7]) the functors $F \mapsto \varinjlim \Gamma_{Z \setminus T}(X \setminus T, F|_{X \setminus T})$ and $F \mapsto \coprod_{x \in X^\ell} \varinjlim \Gamma_{U \cap \overline{\{x\}}}(U, F|_U)$ are left exact; the homomorphism (3.4) for $i > 0$ is obtained by deriving these two functors.

Lemma 3.2.35. *The homomorphism (3.4) is an isomorphism.*

Proof. We begin with the following general observation: Let (T, Z) be a pair of closed subsets of X such that $\operatorname{codim} T > \ell$, $\operatorname{codim} Z = \ell$, and denote by $Z_1, \dots, Z_m, Z'_1, \dots, Z'_n$ the irreducible components of $Z \setminus T$, where $\operatorname{codim} Z_1 = \cdots \operatorname{codim} Z_m = \ell$ and $\operatorname{codim} Z'_1 = \cdots = \operatorname{codim} Z'_n > \ell$; now, let U be an open subset of X intersecting all the irreducible components of Z_j but none of the irreducible components Z'_j, then if we set $T' := (Z \setminus U) \cup T$, which is closed and of codimension $> \ell$, we have $U \cap Z = Z \setminus T'$ so that by the excision theorem (Th. 3.2.7) we have a canonical isomorphism $H^i_{Z \cap U}(U, F|_U) \cong H^i_{Z \setminus T'}(X \setminus T', F|_{X \setminus T'})$ for all $i \geq 0$. Such a subset U may always be found; in fact, it may be chosen such that the sets $U \cap Z_i$ are pairwise disjoint by taking e.g. the complement of $(\bigcup_{j \neq k} Z_j \cap Z_k) \cup (\bigcup_j Z'_j) \cup T$ in X.

Now we show that the homomorphism is injective. Let s be an element of the kernel of (3.4), then there exist a pair (T, Z) of closed subsets of X with $\operatorname{codim} T > \ell$, $\operatorname{codim} Z = \ell$ and an element $t \in H^i_{Z \setminus T}(X \setminus T, F|_{X \setminus T})$ such that t gets mapped to s (see [KS06, Prop. 3.1.3. & Cor. 3.1.5]). We denote by $Z_1, \dots, Z_n$ the irreducible components of $Z \setminus T$ of codimension ℓ, then by the observation at the beginning of this proof there exists a closed subset T' of codimension $> \ell$ containing T such that the sets $Z_i \setminus T'$ are non-empty and pairwise disjoint; in fact, by the same token we may chose T' sufficiently large such that $s|_{X \setminus T'} = 0$ and thus $s = 0$.

Finally, to see surjectivity let $x \in X^\ell$ and U an open neighbourhood of x, then, if we set $T := \overline{\{x\}} \setminus U$, we see that $H^i_{U \cap \overline{\{x\}}}(U, F|_U)$ is canonically isomorphic to $H^i_{\overline{\{x\}} \setminus T}(X \setminus T, F|_{X \setminus T})$ by the excision theorem (Theorem 3.2.7). Thus any element in $\varinjlim H^i_{U\cap\overline{\{x\}}}(U, F|_U)$ and therefore any element in $\coprod_{x\in X^\ell} H^i_{U\cap\overline{\{x\}}}(U, F|_U)$ is in the image of (3.4). □

For for all $i, \ell \geq 0$ we have a canonical homomorphism

$$\varinjlim_{\substack{T \text{ closed} \subseteq X: \\ \operatorname{codim} T = \ell+1}} H^i(X \setminus T, F|_{X\setminus T}) \to \varinjlim_{\substack{T \text{ closed} \subseteq X: \\ \operatorname{codim} T > \ell}} H^i(X \setminus T, F|_{X\setminus T}) \tag{3.6}$$

induced by the inclusion

$$\left\{ \; T \subseteq X \;\middle|\; T \text{ closed, } \operatorname{codim} T = \ell + 1 \; \right\} \subseteq \left\{ \; T \subseteq X \;\middle|\; T \text{ closed, } \operatorname{codim} T > \ell \; \right\},$$

which is an isomorphism by [KS06, Prop. 2.5.2].
Similarly, we obtain a homomorphsim

$$\varinjlim_{\substack{T,Z \text{ closed} \subseteq X: \\ \operatorname{codim} T > \ell \\ \operatorname{codim} Z = \ell}} H^i(X \setminus (T \cup Z), F|_{X\setminus(T\cup Z)}) \to \varinjlim_{\substack{Z \text{ closed} \subseteq X: \\ \operatorname{codim} Z = \ell}} H^i(X \setminus Z, F|_{X\setminus Z}) \tag{3.7}$$

induced by the order preserving map

$$\begin{array}{rcl} \left\{ \; (T, Z) \subseteq X \times X \;\middle|\; \begin{array}{c} T, Z \text{ closed,} \\ \operatorname{codim} T > \ell, \operatorname{codim} Z = \ell \end{array} \right\} & \to & \left\{ \; Z \subseteq X \;\middle|\; Z \text{ closed, } \operatorname{codim} Z = \ell \; \right\} \\ (T, Z) & \mapsto & T \cup Z \end{array}$$

(recall that the empty set has codimension ∞), which is again an isomorphism by [KS06, Prop. 2.5.2].

Notation 3.2.36. For any $i, \ell \in \mathbb{N}$ we write

$$H^i(X_\ell) := \varinjlim_{\substack{T \text{ closed} \subseteq X: \\ \operatorname{codim} T = \ell+1}} H^i(X \setminus T, F|_{X\setminus T}).$$

⌟

Lemma 3.2.37. *For all $i \geq 0$ we have $H^i(X_{-1}, F) \cong 0$.*

Proof. The lemma is a corollary of the following result: Let $\mathcal{I}$ be a small category containing a final object, and let $\mathcal{C}$ be a category, then for any functor $A : \mathcal{I} \to \mathcal{C}$ the colimit of A exists and is isomorphic to $A(1)$. This is easily check by hand; alternatively, one could apply [KS06, Prop. 2.5.2] to see that the inclusion functor $\{1\} \hookrightarrow \mathcal{I}$ is cofinal. The final object in the category of all closed subsets of X of codimension 0 is X itself so we have $H^i(X_{-1}, F) \cong H^i(X \setminus X, F) \cong H^i(\varnothing, F,) \cong 0$. □

Constructing the complex associated to F

We begin by constructing the complex of global sections of the complex associated to F. For any closed subsets T and Z of X we obtain the long exact sequence

$$0 \longrightarrow H^0_Z(X \setminus T, F) \longrightarrow H^0(X \setminus T, F) \longrightarrow H^0(X \setminus (T \cup Z), F) \longrightarrow H^1_Z(X \setminus T, F) \longrightarrow \cdots$$

by Proposition 3.2.8. By taking the filtered colimit of such long exact sequences over closed subsets $T, Z \subseteq X$ such that $\operatorname{codim} Z = \ell$, $\operatorname{codim} T > \ell$ we then obtain a long exact sequence

$$0 \longrightarrow \coprod_{x \in X^\ell} \varinjlim H^0_{U \cap \overline{\{x\}}}(U, F|_U) \longrightarrow H^0(X_\ell, F) \longrightarrow H^0(X_{\ell-1}, F) \longrightarrow \coprod_{x \in X^\ell} \varinjlim H^1_{U \cap \overline{\{x\}}}(U, F|_U) \longrightarrow \cdots \tag{3.8}$$

via the isomorphisms (3.4), (3.6), (3.7) (recall that filtered colimits are exact in **Ab** (see [KS06, Cor. 3.1.7])). By Lemma 3.2.37 and by setting $\ell = 0$ in (3.8) we see that for all $i \geq 0$:

$$\coprod_{x \in X^0} \varinjlim H^i_{U \cap \overline{\{x\}}}(U, F|_U) \cong H^i(X_0, F). \tag{3.9}$$

There is a homomorphism $\Gamma(X, F) \to H^0(X_0, F)$ given by choosing any closed subset T of codimension 1 and then composing $\Gamma(X, F) \to \Gamma(X \setminus T, F) \to H^0(X_0, F)$; this homomorphism is independent of the choice of T, for if we choose another closed subset T' of codimension 1, then the following diagram commutes:

$$\begin{array}{ccccc}
 & & \Gamma(X \setminus T, F) & & \\
 & \nearrow & \downarrow & \searrow & \\
\Gamma(X, F) & & \Gamma(X \setminus (T \cup T'), F) & \longrightarrow & H^0(X^0, F) \\
 & \searrow & \uparrow & \nearrow & \\
 & & \Gamma(X \setminus T', F). & &
\end{array}$$

Now by splicing together the exact sequences

$$H^0(X_0, F) \to \coprod_{x \in X^1} \varinjlim H^1_{U \cap \overline{\{x\}}}(U, F|_U) \to H^1(X_1, F)$$

$$H^1(X_1, F) \to \coprod_{x \in X^2} \varinjlim H^2_{U \cap \overline{\{x\}}}(U, F|_U) \to H^2(X_2, F)$$

$$\cdots$$

and

$$\Gamma(X, F) \to H^0(X_0, F) \cong \coprod_{x \in X^0} \varinjlim H^0_{U \cap \overline{\{x\}}}(U, F|_U)$$

(see Lemma 3.2.37) we obtain a cochain complex of Abelian groups

$$\Gamma(X, F) \quad \to \quad \coprod_{x \in X^0} \varinjlim H^0_{U \cap \overline{\{x\}}}(U, F|_U) \quad \to \quad \coprod_{x \in X^1} \varinjlim H^1_{U \cap \overline{\{x\}}}(U, F|_U) \quad \to \quad \cdots . \tag{3.10}$$

(To see that $d^2 = 0$ for the first two homomorphisms note that $\Gamma(X, F) \to H^0(X_0, F)$ factors through $H^0(X_1, F) \to H^0(X_0, F)$.)
For every $i \geq 0$ the Abelian group $\coprod_{x \in X^i} \varinjlim H^i_{U \cap \overline{\{x\}}}(U, F|_U)$ is the global section of the presheaf

$$\begin{aligned} F^i : \mathbf{Ouv}_X^{\mathrm{op}} &\to \mathbf{Ab} \\ V &\mapsto \coprod_{x \in V^i} \varinjlim_{\substack{U \ni x \\ \text{open in } V}} H^i_{U \cap \overline{\{x\}}}(U, F|_U). \end{aligned} \tag{3.11}$$

Just as in Example 3.2.33 we see that the presheaves F^i are flasque sheaves by noting that they are canonically isomorphic to the coproduct of skyscraper sheaves $\coprod_{x \in X^i} i_x(\varinjlim H^i_{U \cap \overline{\{x\}}}(U, F|_U))$, where for every $x \in X$ the map $i_x : \{x\} \hookrightarrow X$ denotes the canonical inclusion. Repeating the construction of (3.10) for all open subsets $V \subseteq X$ we obtain a complex of sheaves which which we denote by

$$F^0 \to F^1 \to F^2 \to \cdots \tag{3.12}$$

together with an augmentation map $F \to F^0$.

Proposition 3.2.38. *The above construction is functorial and additive in F.*

Proof. Let F', F'' be Abelian sheaves on X and let $\varphi : F' \to F''$ be a morphism, then functoriality follows from the commutativity of

$$\begin{array}{ccccc}
H^i(X_i, F') & \longrightarrow & \coprod\limits_{x\in X^{i+1}} \varinjlim H^{i+1}_{U\cap\overline{\{x\}}}(U, F'|_U) & \longrightarrow & H^{i+1}(X_{i+1}, F') \\
\downarrow & & \downarrow & & \downarrow \\
H^i(X_i, F'') & \longrightarrow & \coprod\limits_{x\in X^{i+1}} \varinjlim H^{i+1}_{U\cap\overline{\{x\}}}(U, F''|_U) & \longrightarrow & H^{i+1}(X_{i+1}, F'')
\end{array}$$

for $i \geq 1$ and

$$\begin{array}{ccc}
\Gamma(X, F') & \longrightarrow & H^0(X_0, F') \cong \coprod_{x\in X^0} \varinjlim H^0_{U\cap\overline{\{x\}}}(U, F'|_U) \\
\downarrow & & \downarrow \\
\Gamma(X, F'') & \longrightarrow & H^0(X_0, F'') \cong \coprod_{x\in X^0} \varinjlim H^0_{U\cap\overline{\{x\}}}(U, F''|_U),
\end{array}$$

which in turn follows from the commutativity of

$$\begin{array}{ccccc}
H^i(X \setminus (T\cup Z), F') & \longrightarrow & H^{i+1}_Z(X\setminus T, F') & \longrightarrow & H^{i+1}(X\setminus T, F') \\
\downarrow & & \downarrow & & \downarrow \\
H^i(X \setminus (T\cup Z), F'') & \longrightarrow & H^{i+1}_Z(X\setminus T, F'') & \longrightarrow & H^{i+1}(X\setminus T, F')
\end{array}$$

and

$$\begin{array}{ccc}
\Gamma(X, F') & \longrightarrow & H^0(X\setminus T, F') \\
\downarrow & & \downarrow \\
\Gamma(X, F'') & \longrightarrow & H^0(X\setminus T, F'')
\end{array}$$

for all $i \geq 0$ and all closed subsets $T, Z \subseteq X$.
Additivity follows from the additivity of local cohomology functors, filtered colimits (see [KS06, Cor. 3.1.7]) and coproducts. ☐

Before we can prove that the sequence $F \to F^0 \to F^1 \to \cdots$ is exact when F is Cohen-Macaulay we need Lemma 3.2.40 which in turn relies on the following lemma.

Lemma 3.2.39. *Let $\mathcal{A}$, $\mathcal{B}$ be Abelian categories, $F, G : \mathcal{A} \to \mathcal{B}$ left exact functors and $\eta : F \Rightarrow G$ a natural transformation. Assume that $\mathcal{A}$ has enough injective objects, then η is an isomorphism iff its restriction to the full subcategory of $\mathcal{A}$ spanned by injective objects is an isomorphism.*

Proof. Let $X \in \mathcal{A}$ and let I^* be an injective resolution of X, then we obtain the commutative diagram

$$
\begin{array}{ccc}
F(I^2) & \xrightarrow{\cong} & G(I^2) \\
\uparrow & & \uparrow \\
F(I^1) & \xrightarrow{\cong} & G(I^1) \\
\uparrow & & \uparrow \\
F(X) & \longrightarrow & G(X),
\end{array}
\tag{3.13}
$$

and $\eta_X : F(X) \to G(X)$ must be an isomorphism by the universal property of kernels. □

Lemma 3.2.40. *Let $z \in X$, then the colimit of the sequence*

$$
0 \longrightarrow \coprod_{x \in V^\ell} \varinjlim_{\substack{U \ni x \\ \textit{open in } V}} H^0_{U \cap \overline{\{x\}}}(U, F|_U) \longrightarrow H^0(V_\ell, F) \longrightarrow H^0(V_{\ell-1}, F) \longrightarrow \coprod_{x \in V^\ell} \varinjlim_{\substack{U \ni x \\ \textit{open in } V}} H^1_{U \cap \overline{\{x\}}}(U, F|_U) \longrightarrow \cdots
$$

constructed as in (3.8), where V ranges of over all open neighbourhoods of z, is canonically isomorphic to

$$
0 \longrightarrow \coprod_{x \in X_z^\ell} \varinjlim_{\substack{U \ni x \\ \textit{open in } X_z}} H^0_{U \cap \overline{\{x\}}}(U, j^{-1}F|_U) \longrightarrow H^0((X_z)_\ell, j^{-1}F) \longrightarrow H^0((X_z)_{\ell-1}, j^{-1}F), \longrightarrow \coprod_{x \in X_z^\ell} \varinjlim_{\substack{U \ni x \\ \textit{open in } X_z}} H^1_{U \cap \overline{\{x\}}}(U, j^{-1}F|_U) \longrightarrow \cdots
$$

where $j : X_z \hookrightarrow X$ denotes the canonical inclusion.

Proof. For every $\ell \geq 0$ we will construct canonical homomorphisms

$$
\varinjlim_{V \ni z} \coprod_{x \in V^\ell} \varinjlim_{\substack{U \ni x \\ \text{open in } V}} \Gamma_{U \cap \overline{\{x\}}}(U, F|_U) \to \coprod_{x \in X_z^\ell} \varinjlim_{\substack{U \ni x \\ \text{open in } X_z}} \Gamma_{U \cap \overline{\{x\}}}(U, j^{-1}F|_U) \tag{3.14}
$$

and

$$
\varinjlim_{V \ni z} \Gamma(V_\ell, F|_V) \to \Gamma((X_z)_\ell, j^{-1}F), \tag{3.15}
$$

and show that the diagram

$$
\begin{array}{ccccccc}
0 \longrightarrow & \varinjlim_{V\ni z} \coprod_{x\in V^\ell} \varinjlim \Gamma_{U\cap\overline{\{x\}}}(U,F|_U) & \longrightarrow & \varinjlim_{V\ni z} \Gamma(V^\ell, F|_V) & \longrightarrow & \varinjlim_{V\ni z} \Gamma(V_{\ell-1}, F|_V) \\
\downarrow & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \coprod_{x\in X_z^\ell} \varinjlim \Gamma_{U\cap\overline{\{x\}}}(U,j^{-1}F|_U) & \longrightarrow & \Gamma((X_z)_\ell, j^{-1}F|_U) & \longrightarrow & \Gamma((X_z)_{\ell-1}, j^{-1}F|_U)
\end{array}
\tag{3.16}
$$

commutes. The isomorphism between the two long exact sequences in the statement of the lemma is then obtained by plugging in an injective resolution of F and taking cohomology.

Construction of (3.14). Let $\ell \geq 0$. As X_z is the intersection of all open subsets of X containing z we see that

$$\varinjlim_{V\ni z} \coprod_{x\in V^\ell} \varinjlim_{\substack{U\ni x\\ \text{open in } X}} \Gamma_{U\cap\overline{\{x\}}}(U,F|_U) \cong \coprod_{x\in X_z^\ell} \varinjlim_{\substack{U\ni x\\ \text{open in } X}} \Gamma_{U\cap\overline{\{x\}}}(U,F|_U),$$

so that it is enough to show that for each $x \in X_z$ we obtain a homomorphism

$$\varinjlim_{\substack{U\ni x\\ \text{open in } X}} \Gamma_{U\cap\overline{\{x\}}}(U,F|_U) \to \varinjlim_{\substack{U\ni x\\ \text{open in } X_z}} \Gamma_{U\cap\overline{\{x\}}}(U,j^{-1}F|_U). \tag{3.17}$$

The groups in (3.17) can be viewed as subgroups of F_x and $j^{-1}F_x$ respectively and we will show that the canonical isomorphism

$$
\begin{array}{rcl}
F_x & \to & j^{-1}F_x \\
{[(s,U)]} & \mapsto & [(s, U\cap X_z, U)].
\end{array}
\tag{3.18}
$$

restricts to the desired homomorphism; here we express $j^{-1}F_x$ as a quotient of

$$\left\{ (s,U,V) \;\middle|\; x \in V \overset{\text{open}}{\subseteq} X_z,\ V \subseteq U \overset{\text{open}}{\subseteq} X,\ s \in F(U) \right\}.$$

We must show that the image of $\varinjlim \Gamma_{U\cap\overline{\{x\}}}(U,F|_U)$ under (3.18) lies in $\varinjlim \Gamma_{U\cap\overline{\{x\}}}(U,j^{-1}F|_U)$. Let $[(s,U)] \in \varinjlim \Gamma_{U\cap\overline{\{x\}}}(U,F|_U)$, then $s|_{U\setminus\overline{\{x\}}} = 0$. To see now that $[(s,U\cap X_z,U)]$ lies in $\varinjlim \Gamma_{U\cap\overline{\{x\}}}(U,j^{-1}F|_U)$ it is enough to show that there exists an open subset $W \subseteq X$ containing $U \cap X \setminus \overline{\{x\}}$ such that $s|_W = 0$, so we may simply set $W := U \setminus \overline{\{x\}}$.

Proof that (3.14) is an isomorphism. Injectivity is automatically satisfied, as (3.14) is the restriction of an injective map. To show that every element in $\varinjlim \Gamma_{U\cap\overline{\{x\}}}(U,j^{-1}F|_U)$ comes from an element in $\varinjlim \Gamma_{U\cap\overline{\{x\}}}(U,F|_U)$ we note that both the functors $F \mapsto \varinjlim \Gamma_{U\cap\overline{\{x\}}}(U,F|_U)$ and $F \mapsto \varinjlim \Gamma_{U\cap\overline{\{x\}}}(U,j^{-1}F|_U)$ are left exact (see [KS06, Cor. 3.1.7]), so that by Lemma 3.2.39 we may assume that F is injective and thus flasque. In this case $j^{-1}F$ is also flasque by Proposition 3.1.2, and thus also $\underline{\Gamma}_{\overline{\{x\}}} j^{-1}F$ by [Har67, Lm. 1.6]. Any element in $\varinjlim \Gamma_{U\cap\overline{\{x\}}}(U,j^{-1}F|_U) \cong \varinjlim \underline{\Gamma}_{\overline{\{x\}}} j^{-1}F(U)$ may thus be represented by a triple of the form (s,X,X_z). By assumption there exists an open subset $W \subseteq X$ containing $X_z \setminus \overline{\{x\}}$ such that $s|_W = 0$; from this we deduce that $\operatorname{supp} s \cap X_z \subseteq \overline{\{x\}} \cap X_z$. Denote by $z_1, \dots, z_n$ those generic points of the irreducible components of $\operatorname{supp} s$ which lie outside of X_z, then setting $U := X \setminus (\overline{\{z_1\}} \cup \cdots \cup \overline{\{z_n\}})$ we have $s|_{U \setminus \overline{\{x\}}} = 0$. It is clear by construction that $[(s|_U, U)]$ is mapped to $[(s, X, X_z)]$ by (3.18).

Construction of (3.15). First we observe that we have the following isomorphisms:

$$\begin{aligned} \varinjlim_{V \ni z} \Gamma(V_\ell, F|_V) &= \varinjlim_{V \ni z} \varinjlim_{\substack{T \subseteq V \\ \operatorname{codim} T = \ell+1}} \Gamma(V \setminus T, F) \\ &\cong \varinjlim_{V \ni z} \varinjlim_{\substack{T \subseteq X \\ \operatorname{codim} T = \ell+1}} \Gamma(V \setminus T, F) \end{aligned}$$

and

$$\begin{aligned} \Gamma((X_z)_\ell, j^{-1}F) &= \varinjlim_{\substack{T \subseteq X_z^\ell \\ \operatorname{codim} T = \ell+1}} \varinjlim_{U \supseteq X_z \setminus T} \Gamma(U, F) \\ &\cong \varinjlim_{\substack{T \subseteq X \\ \operatorname{codim} T = \ell+1}} \varinjlim_{U \supseteq X_z \setminus T} \Gamma(U, F), \end{aligned}$$

which in both cases result from an easy application of [KS06, Prop. 2.5.2]. The homomorphism (3.15) is then given by the universal property of $\varinjlim_{V \ni z} \Gamma(V_\ell, F|_V)$ viewing $\Gamma((X_z)_\ell, j^{-1}F)$ as the vertex of the cone given by the composition of

$$\Gamma(V \setminus T, F) \to \varinjlim_{U \supseteq X_z \setminus T} \Gamma(U, F) \to \Gamma((X_z)_\ell, j^{-1}F)$$

for each open neighbourhood $V \ni z$ in X and every closed subset $T \subseteq X$ of codimension $\ell + 1$. Explicitly we get

$$\begin{aligned} \varinjlim_{V \ni z} \Gamma(V_\ell, F) &\to \Gamma((X_z)_\ell, j^{-1}F) \\ [(s, V, T)] &\mapsto [(s, X_z \setminus T, V \setminus T)]. \end{aligned}$$

which assemble to form the homomorphism (3.15). Before we can proceed to show that (3.15) is an isomorphism we must first show that (3.16) commutes.

Commutativity of (3.16). To see that the central square commutes it is enough to show that for every open neighbourhood $V \ni z$ in X the diagram

$$\begin{array}{ccc} \varinjlim_{V \supseteq U \ni x} \Gamma_{U \cap \overline{\{x\}}}(U, F|_U) & \longrightarrow & \Gamma(V_\ell, F) \\ \downarrow & & \downarrow \\ \varinjlim_{X_z \supseteq U \ni x} \Gamma_{U \cap \overline{\{x\}}}(U, j^{-1}F|_U) & \longrightarrow & \Gamma((X_z)_\ell, F) \end{array}$$

commutes. By the introductory remark of the proof of Lemma 3.2.35 any element in $\varinjlim \Gamma_{U \cap \overline{\{x\}}}(U, F|_U)$ may be represented by an element $s \in \Gamma_{(V \setminus T) \cap \overline{\{x\}}}(V \setminus T, F|_{V \setminus T})$ where $T \subset V$

is a closed set of codimension $\ell+1$. Chasing this element around the above diagram we get

$$\begin{array}{ccc} [(s, V\setminus T)] & \longmapsto & [(s,T)] \\ \downarrow & & \downarrow \\ [(s, X_z\setminus T, V\setminus T)] & \longmapsto & [(s, X_z\setminus T, V\setminus T)]. \end{array}$$

To see that the rightmost square of (3.16) commutes it is enough to show that for every open neighbourhood $V \ni z$ in X the diagram

$$\begin{array}{ccc} \Gamma(V_\ell, F|_V) & \longrightarrow & \Gamma(V_{\ell-1}, F|_V) \\ \downarrow & & \downarrow \\ \Gamma((X_z)_\ell, j^{-1}F) & \longrightarrow & \Gamma((X_z)_{\ell-1}, j^{-1}F) \end{array}$$

commutes. Let $T, Z \subseteq X$ be closed subsets such that $\operatorname{codim} T = \ell+1$ and $\operatorname{codim} Z = \ell$, and let $s \in \Gamma(V\setminus T, F)$, then chasing s through the above diagram we see

$$\begin{array}{ccc} [(s,T)] & \longmapsto & [(s|_{V\setminus(T\cup Z)}, T\cup Z)] \\ \downarrow & & \downarrow \\ [(s, X_z\setminus T, V\setminus T)] & \longmapsto & [(s|_{V\setminus(T\cup Z)}, X_z\setminus(T\cup Z), V\setminus(T\cup Z))]. \end{array}$$

Proof that (3.15) is an isomorphism. We shall prove this by induction. For $\ell = -1$ we have $\varinjlim_{V\ni z}\Gamma(X_\ell, F) \cong \Gamma((X_z)_\ell, F) \cong 0$ so that (3.15) is trivially an isomorphism. Now, assume that (3.15) is an isomorphism for a given $\ell \geq 0$. By Lemma 3.2.39 we may assume that F is injective; in this case we may complete the diagram (3.16) to a morphism of short exact sequences. If we plug $\ell+1$ into (3.16), then the central vertical homomorphism is an isomorphism because the outer vertical homomorphisms are. □

Corollary 3.2.41. *The inverse image under j of the complex associated to F is canonically isomorphic to the complex associated to $j^{-1}F$.* □

Corollary 3.2.42. *The localisation at z of the complex associated to F is given by the global sections of the complex associated to $j^{-1}F$.* □

Theorem 3.2.43. *If F satisfies the Cohen-Macaulay condition, then the complex associated to F is exact (and is thus isomorphic to the Cousin complex of F).*

Proof. Let d denote the dimension of X. For all $\ell \geq 0$ the exact sequences (3.8) split into the exact sequences

$$0 \to H^i(X_\ell, F) \xrightarrow{\cong} H^i(X_{\ell-1}, F) \to 0 \qquad (i > \ell \text{ or } i < \ell-1)$$

and

$$0 \to H^{\ell-1}(X_\ell, F) \to H^{\ell-1}(X_{\ell-1}, F) \to \coprod_{x\in X^\ell} \varinjlim_{x\in U\text{open}} H^\ell_{U\cap\overline{\{x\}}}(U, F|_U) \to H^\ell(X_\ell, F) \to H^\ell(X_{\ell-1}, F) \to 0.$$

From this and Lemma 3.2.37 we see that for all $\ell \geq 0$ we have

$$H^\ell(X_{\ell-1}, F) \cong H^\ell(X_{\ell-2}, F) \cong \cdots \cong H^\ell(X_{-1}, F) \cong 0$$

and

$$H^\ell(X_{\ell+1}, F) \cong H^\ell(X_{\ell+2}, F) \cong \cdots \cong H^\ell(X_{d+1}, F) \cong H^\ell(X, F). \tag{3.19}$$

If we replace X with X_z in the above diagrams we see that the sequences

$$0 \to H^{\ell-1}((X_z)_{\ell-1}, j^{-1}F) \to \coprod_{x \in X_z^\ell} \varinjlim_{x \in U \text{open}} H^\ell_{U \cap \overline{\{x\}}}(U, j^{-1}F|_U) \to H^\ell((X_z)_\ell, j^{-1}F) \to 0.$$

are exact for $\ell \geq 0$ and thus the Cousin complex of F is exact in positive degree. It remains to show that $F_z \cong j^{-1}F(X_z)$ is the kernel of $j^{-1}F^0(X_z) \to j^{-1}F^1(X_z)$; this follows from the fact that the natural homomorphism $j^{-1}F(X_z) \to H^0(X_1 F)$ is an isomorphism, which may be seen by applying (3.19). □

Remark 3.2.44. From the proof of Theorem 3.2.43 one may see explicitly that the cohomology groups of $F(X) \to F^0(X) \to F^1(X) \to \cdots$ are isomorphic to the groups $H^i(X, F)$, by noting that $F(X) \to F^0(X) \to F^1(X) \to \cdots$ is obtained by splicing together the exact sequences

$$0 \to \Gamma(X, F) \to \underbrace{H^0(X_0, F)}_{\cong \coprod_{x \in X^0} \varinjlim H^0_{U \cap \overline{\{x\}}}(U, F|_U)} \to \coprod_{x \in X^1} \varinjlim H^1_{U \cap \overline{\{x\}}}(U, F|_U) \to H^1(X_1, F) \to 0$$

$$0 \to H^1(X, F) \to H^1(X_1, F) \to \coprod_{x \in X^2} \varinjlim H^2_{U \cap \overline{\{x\}}}(U, F|_U) \to H^2(X_2, F) \to 0$$

$$\cdots$$

(keeping in mind that the i-th cohomology of a cochain complex may be computed as the kernel of $\mathrm{Cok}_{d^{i-1}} \to \mathrm{Coim}_{d^i}$). ⌟

Part II

Pro-algebraic Resolutions of Regular Schemes

Chapter 4

Sheaves Associated to Algebraic k-Groups are Cohen-Macaulay

Throughout this chapter k denotes an algebraically closed field of characteristic zero and X denotes a regular k-scheme. All k-groups in this chapter are assumed to be commutative so "k-group" will mean "commutative k-group".
In this chapter we prove that sheaves on X associated to algebraic k-groups are Cohen-Macaulay, so that we obtain a functor $\mathbf{AGS}_k \to \mathbf{Ch}_{\geq 0}(\mathbf{Sh}_X)$ by putting together (2.2) and the functor associating to any Cohen-Macaulay sheaf on X its Cousin resolution.
We proceed in two steps corresponding to §4.1 & 4.2 respectively. Let G be an algebraic k-group; in §4.1 we show that proving that G_X is Cohen-Macaulay reduces to proving that the cohomology groups $H^i_{\{x\}}(\operatorname{Spec} \mathscr{O}_{X,x}, G)$ vanish for all $i \neq \dim \operatorname{Spec} \mathscr{O}_{X,x}$. In §4.2 we verify this assertion by showing that for every regular local ring $(A, \mathfrak{m})$ the groups $H^i_{\{\mathfrak{m}\}}(\operatorname{Spec} A, G)$ vanish for $i \neq \dim A$.

4.1 Reduction to local cohomology groups of regular local rings

Lemma 4.1.1. *Let A be a ring and $S \subseteq A$ a multiplicatively closed subset, and denote by $i : \operatorname{Spec} S^{-1}A \hookrightarrow \operatorname{Spec} A$ the canonical inclusion, then the diagram*

$$\begin{array}{ccc} \mathbf{QCoh}_{\mathscr{O}_{\operatorname{Spec} A}} & \xrightarrow{i^*} & \mathbf{QCoh}_{\mathscr{O}_{\operatorname{Spec} S^{-1}A}} \\ \downarrow & & \downarrow \\ \mathbf{PSh}_{\mathscr{O}_{\operatorname{Spec} A}} & \xrightarrow{i^\dagger} & \mathbf{PSh}_{\mathscr{O}_{\operatorname{Spec} S^{-1}A}} \end{array}$$

commutes up to canonical isomorphism.

Sketch of proof. Throughout this proof we will only consider sheaves defined on the canonical basis of any affine scheme, as these correspond uniquely to sheaves on the whole topology (see e.g. [EGA I, §**0**.3.2]).
We begin with a few preparations. First we note that if $S \subseteq T \subseteq A$ are multiplicatively closed

subsets, then there is a unique homomorphism of rings $S^{-1}A \to T^{-1}A$ making the triangle

$$\begin{array}{ccc} S^{-1}A & \longrightarrow & T^{-1}A \\ & \nwarrow \quad \nearrow & \\ & A & \end{array} \tag{4.1}$$

commute. There is a maximal multiplicatively closed subset T containing S such that the horizontal homomorphism in (4.1) is an isomorphism; this multiplicative subset is called the *saturation of* S, and a multiplicative subset is called *saturated* if it is equal to its saturation. We will need a further characterisation of saturated multiplicative subsets: A multiplicative subset is saturated iff it is the complement of the union of prime ideals (see [Kap74, Th. 2]). We thus see that there is bijection between the saturated multiplicatively closed subsets of A and the subsets of $\operatorname{Spec} A$ which constitute the image of $\operatorname{Spec} S^{-1}A \to \operatorname{Spec} A$ for some multiplicatively closed subset S. We denote by $\mathbf{Mult}_{\operatorname{Spec} A}$ the category consisting of such subsets of $\operatorname{Spec} A$ together with inclusions. We may now define the contravariant functor

$$\begin{array}{rcl} \mathbf{Mult}^{\mathrm{op}}_{\operatorname{Spec} A} & \to & \mathbf{Ring} \\ \mathrm{Im}_{\operatorname{Spec} S^{-1}A \to \operatorname{Spec} A} & \mapsto & S^{-1}A, \end{array} \tag{4.2}$$

where S is saturated, and where morphisms between the rings $S^{-1}A$ are obtained as in (4.1). These morphisms may also be viewed slightly differently: It is easily seen that the saturated multiplicative subsets of $S^{-1}A$ are in canonical bijection to the saturated multiplicative subsets of A containing S; the morphism $S^{-1}A \to T^{-1}A$ is the localisation of $S^{-1}A$ with respect to the multiplicative subset of $S^{-1}A$ corresponding to T. We thus see that for any multiplicative subset S the basic open subsets of $\operatorname{Spec} S^{-1}A$ are objects in $\mathbf{Mult}_{\operatorname{Spec} A}$.
Similarly as for A, for every A-module we obtain a contravariant functor

$$\begin{array}{rcl} \mathbf{Mult}^{\mathrm{op}}_{\operatorname{Spec} A} & \to & \mathbf{Ab} \\ \mathrm{Im}_{\operatorname{Spec} S^{-1}A \to \operatorname{Spec} A} & \mapsto & S^{-1}M, \end{array} \tag{4.3}$$

where S is again saturated. For every saturated multiplicative subset S the group $S^{-1}M$ comes canonically equipped with the structure of an $S^{-1}A$-module; the collection of these $S^{-1}A$-module structures for saturated multiplicative subsets S is compatible with restriction; one could reasonably speak of a "module over the functor (4.2)". Finally, we note that given any further A-module N, a homomorphism of A-modules $M \to N$ induces a homomorphism between their corresponding modules over (4.2).
After these preparations proving the lemma is easy. Let $S \subseteq A$ be a saturated multiplicatively closed subset, and let i be as in the statement of the lemma. We first discuss i^*: It is clear by the characterisation of i^* in terms of A- and $S^{-1}A$-modules (see e.g. [EGA I, Cor. I.1.7.7]) that for an A-module M the $\mathscr{O}_{\operatorname{Spec} S^{-1}A}$-module $i^*(\tilde{M})$ is isomoprhic to the $\mathscr{O}_{\operatorname{Spec} S^{-1}A}$-module corresponding to the contravariant functor given by restricting (4.3) to the basic open subsets of $\operatorname{Spec} S^{-1}A$ together with the action of the matching restriction of (4.2). We now discuss $i^\dagger$: Any object in $\mathbf{Mult}_{\operatorname{Spec} A}$ is the intersection of basic open subsets of $\operatorname{Spec} A$; this may be seen

by the canonical isomorphism $\varinjlim_{t\in T} A_t \cong T^{-1}A$ for any multiplicatively closed subset T. Now, let U denote a basic open subset of $\operatorname{Spec} S^{-1}A$, and denote by T the corresponding saturated multiplicatively closed subset of A. We have $\Gamma(i^\dagger \tilde{M}, U) \cong \varinjlim_{t\in T} M_t \cong T^{-1}M$, and we see that the functor $\mathbf{QCoh}_{\mathscr{O}_{\operatorname{Spec} S^{-1}A}} \to \mathbf{PSh}_{\mathscr{O}_{\operatorname{Spec} S^{-1}A}}$ simply corresponds to forgetting the action of the restriction of (4.2) to $\operatorname{Spec} S^{-1}A$. □

Proposition 4.1.2. *Let $x \in X$, then for every algebraic k-group G and every $i \geq 0$ the canonical homomorphism (see (3.1))*

$$\varinjlim_{U\ni x} H^i_{U\cap\overline{\{x\}}}(U, G) \to H^i_{\{x\}}(\operatorname{Spec} \mathscr{O}_{X,x}, G) \tag{4.4}$$

is an isomorphism.

Proof. First we show that if there exist algebraic k-groups G', G'' such that G fits into a short exact sequence

$$0 \to G' \to G \to G'' \to 0$$

which induces a short exact sequence of Zariski sheaves, then the statement of the proposition for G follows from the statement of the proposition for G' and G''. From the short exact sequence

$$0 \to G'_X \to G_X \to G''_X \to 0,$$

we obtain the diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & \varinjlim_{U\ni x} H^0_{U\cap\overline{\{x\}}}(U, G') & \longrightarrow & \varinjlim_{U\ni x} H^0_{U\cap\overline{\{x\}}}(U, G) & \longrightarrow & \varinjlim_{U\ni x} H^0_{U\cap\overline{\{x\}}}(U, G'') & \longrightarrow & \varinjlim_{U\ni x} H^1_{U\cap\overline{\{x\}}}(U, G) & \longrightarrow \cdots \\
\downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & H^0_{\{x\}}(\operatorname{Spec} \mathscr{O}_{X,x}, G') & \longrightarrow & H^0_{\{x\}}(\operatorname{Spec} \mathscr{O}_{X,x}, G) & \longrightarrow & H^0_{\{x\}}(\operatorname{Spec} \mathscr{O}_{X,x}, G'') & \longrightarrow & H^1_{\{x\}}(\operatorname{Spec} \mathscr{O}_{X,x}, G') & \longrightarrow \cdots .
\end{array}$$

We may then apply the five lemma to see that the homomorphism (4.4) is an isomorphism for G and for all $i \geq 0$.

By the canonical composition series (2.1) and Propositions 2.4.5 - 2.4.7 we see that it is enough to check the following four cases: The k-group G is an étale k-group, an Abelian variety, isomorphic to $\mathbb{G}_a$ or isomorphic to $\mathbb{G}_m$. It is also clear that w.l.o.g. we may assume that X is connected; we denote its generic point by η.

Case 1: G is an étale k-group. As in this case G_X is flasque (see Proposition 2.4.2), (4.4) is trivially an isomorphism for $i > 0$, so we only need to consider the case $i = 0$. We check the two cases $\dim \mathscr{O}_{X,x} = 0$ and $\dim \mathscr{O}_{X,x} > 0$ separately, which correspond to the cases $x = \eta$ and $x \neq \eta$. Assume that G corresponds to the discrete group Γ. If $x = \eta$, then we have $\varinjlim \Gamma_{U\cap\overline{\{x\}}}(U, G) = \varinjlim \Gamma(U, G) \cong \Gamma \cong \Gamma(\operatorname{Spec} \mathscr{O}_{X,x}, G) = \Gamma_{\{x\}}(\operatorname{Spec} \mathscr{O}_{X,x}, G)$. If $x \neq \eta$, then $\Gamma_{\{x\}}(\operatorname{Spec} \mathscr{O}_{X,x}, G) \cong 0$ and for all open neighbourhoods $U \ni x$ we have $\Gamma_{U\cap\overline{\{x\}}}(U, G) \cong 0$ by Example 3.2.3.

Case 2: G is an Abelian k-variety. As in Case 1, the k-group G_X is flasque (see Proposition 2.4.3) so that we need only consider local cohomology groups of degree 0. We assume first that $x = \eta$. By Theorem 2.3.22 and [GW10, Cor. 9.9] the canonical inclusion of any open subscheme $U \hookrightarrow X$ induces an isomorphism $\Gamma(X, G_X) \xrightarrow{\cong} \Gamma(U, G_X)$, so it is enough to show that the canonical morphism $\operatorname{Spec} K(X) \to X$ induces an isomorphism $\mathbf{Sch}_k(X, G) \xrightarrow{\cong} \mathbf{Sch}_k(\operatorname{Spec} K(X), G)$.

We first show injectivity. Consider two morphisms $\varphi, \psi : X \to G$ such that their restriction to $\operatorname{Spec} K(X)$ coincides. Consider an affine neighbourhood V of $\varphi(\operatorname{Spec} K(X)) = \psi(\operatorname{Spec} K(X))$, isomorphic to $\operatorname{Spec} B$ for some finite type k-algebra B, then there exists an affine neighbourhood U of $\operatorname{Spec} K(X)$, isomorphic to $\operatorname{Spec} A$ for some k-algebra A, such that $\varphi(U), \psi(U) \subseteq V$. We then obtain the diagram

$$K(X) \cong \operatorname{Frac} A \longleftarrow A \underset{\mathscr{O}(\psi|_V)}{\overset{\mathscr{O}(\varphi|_V)}{\leftleftarrows}} B;$$

as $A \hookrightarrow \operatorname{Frac} A$ is injective, we see that $\mathscr{O}(\varphi|_V)$ and $\mathscr{O}(\psi|_V)$, and thus $\varphi|_V$ and $\psi|_V$ agree, and therefore also φ and ψ.

We now show surjectivity. Let $\varphi : \operatorname{Spec} K(X) \to G$ be a morphism, and consider an affine neighbourhood U of $\operatorname{Spec} K(X)$, which corresponds to some k-algebra A, and an affine open neighbourhood V of $\varphi(\operatorname{Spec} K(X))$, which is isomorphic to $\operatorname{Spec} B$ for some finite type k-algebra B. The morphism $\varphi : \operatorname{Spec} K(X) \to V$ then corresponds to homomorphism $B \to K(X)$; the image of this morphism is generated by finitely many elements in $K(X)$, say $f_1/g_1, \ldots, f_n/g_n$. We then see that $B \to K(X)$ factors through $A_{g_1 \cdots g_n}$, so that $\varphi : \operatorname{Spec} K(X) \to V$ is the restriction of a morphism $\operatorname{Spec} A_{g_1 \cdots g_n} \to G$, which in turn corresponds to a unique morphism $X \to G$.

If then $x \neq \eta$, then the isomorphism is trivial, just as in the previous case.

Case 3: G is isomorphic to $\mathbb{G}_a$. Let $\operatorname{Spec} A$ be an affine neighbourhood of x and $\mathfrak{p}$ the prime ideal corresponding to x. Let M be a module over A, then we obtain canonical homomorphisms

$$\Gamma_{\overline{\{x\}}}(\operatorname{Spec} A, \tilde{M})_f \to \Gamma_{D_f \cap \overline{\{x\}}}(D_f, \tilde{M})$$

for any $f \in A$ and

$$\Gamma_{\overline{\{x\}}}(\operatorname{Spec} A, \tilde{M})_{\mathfrak{p}} \to \Gamma_{\{x\}}(\operatorname{Spec} \mathscr{O}_{X,x}, \widetilde{(M_{\mathfrak{p}})}),$$

which are clearly isomorphisms. We thus obtain isomorphisms

$$\begin{aligned} \varinjlim_{f \in A \setminus \mathfrak{p}} \Gamma_{D_f \cap \overline{\{x\}}}(D_f, \tilde{M}|_{D_f}) &\cong \varinjlim_{f \in A \setminus \mathfrak{p}} \Gamma_{\overline{\{x\}}}(\operatorname{Spec} A, \tilde{M})_f \\ &\cong \Gamma_{\overline{\{x\}}}(\operatorname{Spec} A, \tilde{M})_{\mathfrak{p}} \\ &\cong \Gamma_{\{x\}}(\operatorname{Spec} \mathscr{O}_{X,x}, \widetilde{(M_{\mathfrak{p}})}), \end{aligned}$$

whose composition coincides with the canonical homomorphism induced by $\Gamma_{D_f \cap \overline{\{x\}}}(D_f, \tilde{M}|_{D_f}) \to \Gamma_{\{x\}}(\operatorname{Spec} \mathscr{O}_{X,x}, \widetilde{(M_{\mathfrak{p}})})$ for all $f \in A \setminus \mathfrak{p}$ by the previous lemma. By taking an injective resolution of the sheaf associated to M, applying the left exact functors $N \mapsto \varinjlim \Gamma_{U \cap \overline{\{x\}}}(U, \tilde{N}|_U)$ and $N \mapsto \Gamma_{\{x\}}(\operatorname{Spec} \mathscr{O}_{X,x}, \widetilde{(N_{\mathfrak{p}})})$ (filtered colimits of Abelian groups and localisation are exact; see [KS06, Cor. 3.1.7] and [Bou61, Th. II.2.4.1] respectively) to the injective resolution, and taking cohomology of cochain complexes we conclude that

$$\varinjlim_{U \ni x} H^i_{U \cap \overline{\{x\}}}(U, \tilde{M}|_U) \cong H^i_{\{x\}}(\operatorname{Spec} \mathscr{O}_{X,x}, \widetilde{(M_{\mathfrak{p}})})$$

for all $i \geq 0$.

Case 4: G is isomorphic to $\mathbb{G}_m$. Denote by j the canonical morphism $\operatorname{Spec} \mathscr{O}_{X,x} \hookrightarrow X$. It is easily seen that $j^{-1}\mathscr{O}_X^\times \cong \mathscr{O}^\times_{\operatorname{Spec} \mathscr{O}_{X,x}}$. By Example 3.2.33 we know that both $\mathscr{O}_X^\times$ and $\mathscr{O}^\times_{\operatorname{Spec} \mathscr{O}_{X,x}}$ are Cohen-Macaulay. The statement of the theorem then follows from Corollary 3.2.41 and the fact that the Cousin complex associated to a Cohen-Macaulay sheaf is unique up to unique isomorphism (see the introductory discussion of §3.2.3).

Alternatively it is not so difficult to show explicitly that there is a an isomorphism between the Cousin resolution of $\mathscr{O}^\times_{\operatorname{Spec} \mathscr{O}_{X,x}}$ and the inverse image of the Cousin resolution of $\mathscr{O}_X^\times$ under j. $\square$

4.2 The Local Cohomology Groups $H^i_{\{x\}}(X, G)$

Throughout this section $(A, \mathfrak{m})$ denotes a regular local k-algebra with $K := \operatorname{Frac} A$, $X := \operatorname{Spec} A$, $x := \mathfrak{m}$, and G an algebraic k-group.

The goal of this chapter is to show that the cohomology groups $H^i_{\{x\}}(X, G)$ vanish for all $i \geq 0$ except $i = \dim X$. In §4.2.1 we characterise these cohomology groups explicitly for $\dim X = 0, 1$. We then treat the general case in Section 4.2.2, where we reduce to checking different cases of the group G as in Proposition 4.1.2.

4.2.1 The local cohomology groups $H^i_{\{x\}}(X, G)$ for $\dim X = 0, 1$

In both the cases $\dim X = 0, 1$ we will need the following long exact sequence (see Proposition 3.2.8):

$$\begin{array}{ccccccc}
 & \hookrightarrow & H^2_{\{x\}}(X,G) & \longrightarrow & \cdots & & \\
 & \hookrightarrow & H^1_{\{x\}}(X,G) & \longrightarrow & H^1(X,G) & \longrightarrow & H^1(X \setminus \{x\}, G) \\
0 & \longrightarrow & H^0_{\{x\}}(X,G) & \longrightarrow & H^0(X,G) & \longrightarrow & H^0(X \setminus \{x\}, G)
\end{array} \tag{4.5}$$

Theorem 4.2.1. *If* $\dim X = 0$, *then*

$$H^i_{\{x\}}(X,G) \cong \begin{cases} G(A) & \textit{if} \quad i = 0, \\ 0 & \textit{if} \quad i \neq 0. \end{cases}$$

Proof. By definition $H^0_{\{x\}}(X,G) = \left\{ f \in \mathbf{Sch}_k(X,G) \;\middle|\; \operatorname{supp} f = \{x\} \right\} = \mathbf{Sch}_k(X,G) = G(A)$. For $i \geq 0:\ H^i(X \setminus \{x\}, G) = H^i(\varnothing, G) \cong 0$, and for $i > 0:\ H^i(X,G) \cong 0$ by Theorem 3.1.11; applying these two results to the exact sequence (4.5) yields that $H^i_{\{x\}}(X,G) \cong 0$ for $i > 0$. $\square$

Theorem 4.2.2. *If* $\dim X = 1$*, then*

$$H^i_{\{x\}}(X,G) \cong \begin{cases} G(K)/G(A) & \text{if} \quad i = 1, \\ 0 & \text{if} \quad i \neq 1. \end{cases}$$

Proof. By Example **??** $H^0_{\{x\}}(X,G) \cong 0$. By Corollary 3.1.13 and Theorem 3.1.11 $H^i(X,G) \cong 0$ for all $i \geq 1$ and $H^i(X \setminus \{x\}, G) \cong 0$ for all $i \geq 1$; applying these two latter results to the exact sequence (4.5) we see that $H^i_{\{x\}}(X,G) \cong 0$ for all $i \geq 2$. We are then left with the short exact sequence

$$0 \to H^0(X,G) \to H^0(X \setminus \{x\}, G) \to H^1_{\{x\}}(X,G) \to 0. \tag{4.6}$$

Since A is an integral domain of dimension one, the scheme $X \setminus \{x\}$ is integral and has exactly one point, so it is isomorphic to the spectrum of a field. Viewing A and all its localisations as subrings of K we see that $\mathscr{O}_X(X \setminus \{x\}) = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A \setminus \{\mathfrak{m}\}} A_{\mathfrak{p}} = A_{(0)} = K$, so that $X \setminus \{x\}$ is canonically isomorphic to $\operatorname{Spec} K$, and the morphism $X \to X \setminus \{x\}$ corresponds to the homomorphism $A \to K$. Applying this result to the short exact sequence (4.6) yields that $H^1_{\{x\}}(X,G) \cong G(K)/G(A)$. $\square$

4.2.2 The local cohomology groups $H^i_{\{x\}}(X,G)$ for all dimensions

Theorem 4.2.3.

$$H^i_{\{x\}}(X,G) \cong 0 \quad (i \neq \dim X).$$

Proof. First we note that if there exist algebraic k-groups G', G'' such that G fits into a short exact sequence

$$0 \to G' \to G \to G'' \to 0$$

which induces a short exact sequence of Zariski sheaves, then the statement of the proposition for G follows from the statement of the proposition for G' and G''. Indeed, let

$$0 \to G'_X \to G_X \to G''_X \to 0,$$

be the induced short exact sequence of Abelian sheaves, then this is immediately seen by a simple argument using the long exact sequence

$$\begin{array}{l} \hookrightarrow H^1_{\{x\}}(X,G') \longrightarrow \quad \cdots \\ 0 \longrightarrow H^0_{\{x\}}(X,G') \longrightarrow H^0_{\{x\}}(X,G) \longrightarrow H^0_{\{x\}}(X,G'') \end{array} .$$

By the canonical composition series (2.1) and Propositions 2.4.5 - 2.4.7 we see that it is enough to check the following four cases: The k-group G is an étale k-group, an Abelian variety or isomorphic to $\mathbb{G}_a$ or $\mathbb{G}_m$. As both sheaves associated to étale k-groups and Abelian k-varieties are flasque (see Propositions 2.4.2, 2.4.3) these cases reduce to the case when $\dim X = 0$ (see Theorem 4.2.1). If $G \cong \mathbb{G}_m$, then, as $\mathscr{O}_X^\times$ has a flasque resolution of length 1 (see Example 3.2.33), we may reduce to the cases $\dim X = 0, 1$ (see Theorems 4.2.1, 4.2.2). If $G \cong \mathbb{G}_a$, then $G_X \cong \mathscr{O}_X$ so that we may apply Theorem 3.2.22. $\square$

Summarising, by the composition series (2.1) we see that we have a canonical sequence of k-subgroups

$$G_u \hookrightarrow H \hookrightarrow G,$$

where H is the maximal affine k-subgroup of G^0 and G_u is the maximal unipotent subgroup of H. We see now that if $\dim X = 1$, then for all $i \geq 0$

$$H^i_{\{x\}}(X, H) \to H^i_{\{x\}}(X, G)$$

is an isomorphism, and that if $\dim X \geq 2$, then for all $i \geq 0$

$$H^i_{\{x\}}(X, G_u) \to H^i_{\{x\}}(X, G)$$

is an isomorphism.

Chapter 5

The pro-algebraic resolution

Throughout this chapter k denotes an algebraically closed field of characteristic zero. All k-groups in this chapter are assumed to be commutative so "k-group" will mean "commutative k-group".

5.1 Existence of the pro-algebraic resolution

Theorem 5.1.1. *Let X be a connected regular k-scheme, then for every $i \geq 0$ the functor*

$$\begin{array}{rcl} \mathbf{AGS}_k & \to & \mathbf{Ab} \\ G & \mapsto & \Gamma(X, C^i_{X,G}) \end{array}$$

is pro-representable.

Proof. It is enough to show for every regular ring $(A, \mathfrak{m})$ over k that the functor

$$\begin{array}{rcl} \mathbf{AGS}_k & \to & \mathbf{Ab} \\ G & \mapsto & H^{\dim A}_{\{\mathfrak{m}\}}(\operatorname{Spec} A, G) \end{array} \tag{5.1}$$

is pro-representable. Assuming this, denote by $J_{A/k}$ the pro-algebraic k-group representing (5.1), then we see that for every $i \geq 0$ the functor $\mathbf{AGS}_k \to \mathbf{Ab}$, $G \mapsto \Gamma(X, C^i_{X,G})$ is pro-represented by $\prod_{x \in X^i} J_{\mathscr{O}_{X,x}/k}$, for we have canonical isomorphisms

$$\begin{aligned} \mathbf{PAGS}_k\left(\prod_{x \in X^i} J_{\mathscr{O}_{X,x}/k}, G\right) &\cong \bigoplus_{x \in X^i} \mathbf{PAGS}_k(J_{\mathscr{O}_{X,x}/k}, G) \\ &\cong \bigoplus_{x \in X^i} H^i_{\{x\}}(\operatorname{Spec} \mathscr{O}_{X,x}, G), \end{aligned}$$

where the first isomorphism is due the facts that the canonical functor $\mathbf{PAGS}_k \to \widetilde{\mathbf{AGS}}_k$ is fully faithful and that colimits of functors may be obtained by taking colimits objectwise (see [KS06, (2.6.2)]).

We now show that (5.1) is left exact, so that pro-representability follows from Corollary 1.1.11

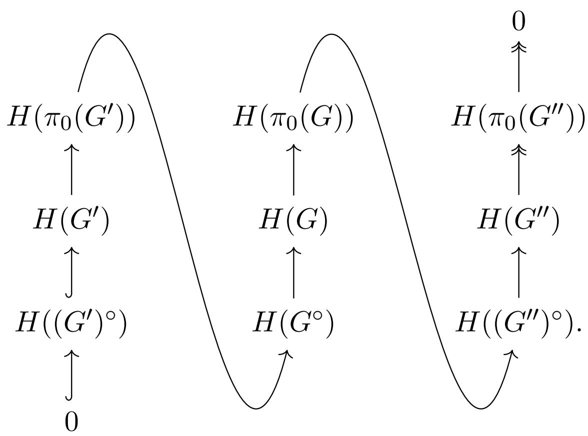
(the category $\mathbf{AGS}_k$ is locally small). We will distinguish between the two cases $\dim A = 0$ and $\dim A > 0$.

Case: $\dim A = 0$. The functor (5.1) is given by $G \mapsto G(A/\mathfrak{m})$ (see Theorem 4.2.1), so that left exactness follows from the fact that the section functors of fppf sheaves are left exact.

Case: $\dim A > 0$. We write $X := \operatorname{Spec} A$, $x := \mathfrak{m}$ and $d := \dim A$. As in several previous proofs we will rely on the composition series (2.1). We first prove that $H^d_{\{x\}}(X, \_)$ is left exact assuming that the functor is left exact on connected algebraic k-groups. Let $0 \to G' \to G \to G'' \to 0$ be a short exact sequence of algebraic k-groups, then we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \hookrightarrow & \pi_0(G') & \longrightarrow & \pi_0(G) & \longrightarrow & \pi_0(G'') & \twoheadrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \hookrightarrow & G' & \hookrightarrow & G & \twoheadrightarrow & G'' & \twoheadrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \hookrightarrow & (G')^\circ & \longrightarrow & G^\circ & \longrightarrow & (G'')^\circ & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & 0. & & \end{array} \tag{5.2}$$

As $H^d_{\{x\}}(X, \_)$ is exact on the columns of (5.2) (see Proposition 2.4.5) and maps the top row to 0 (see Corollary 2.3.3 and Proposition 2.4.2), it is enough to show that the bottom row is mapped to a kernel. By taking homology of the rows of (5.2) we obtain the following long exact sequence:



As $H(G') \cong H(G) \cong H(G'') \cong 0$ we see that $H((G')^\circ) \cong 0$ and $H(\pi_0(G')) \cong H(G^\circ)$. In particular we see that $H(G^\circ)$ is a finite étale k-group (see Proposition 2.3.25). As $(G')^\circ$ is connected the image of $(G')^\circ \to G^\circ$ is connected, so we see that $(G')^\circ$ is canonically isomorphic to the connected component of the kernel of $G^\circ \to (G'')^\circ$. We denote this kernel by K. We then obtain the diagram

$$\begin{array}{ccccccc}
0 & \hookrightarrow & H^d_{\{x\}}(X,(G')^\circ) & \hookrightarrow & H^d_{\{x\}}(X,G^\circ) & \longrightarrow & H^d_{\{x\}}(X,(G'')^\circ) \\
 & & \downarrow{\cong} & & \uparrow & & \\
0 & \hookrightarrow & H^d_{\{x\}}(X,K^\circ) & \xrightarrow{\cong} & H^d_{\{x\}}(X,K) & \twoheadrightarrow & \underbrace{H^d_{\{x\}}(X,\pi_0(K))}_{\cong 0},
\end{array}$$

in which the top row is exact because

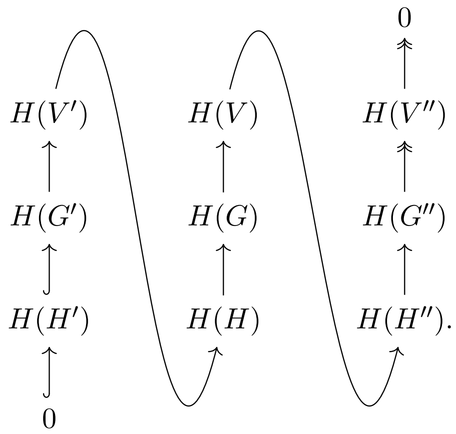
$$0 \to H^d_{\{x\}}(X,K) \to H^d_{\{x\}}(X,G^\circ) \to H^d_{\{x\}}(X,(G'')^\circ)$$

is exact by assumption.

We now prove that $H^d_{\{x\}}(X,\_)$ is left exact on connected algebraic k-groups, assuming it is left exact on connected affine algebraic k-groups. We proceed similarly as before. Let $0 \to G' \to G \to G'' \to 0$ be a short exact sequence of algebraic k-groups, then we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \hookrightarrow & V' & \longrightarrow & V & \longrightarrow & V'' & \twoheadrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \hookrightarrow & G' & \hookrightarrow & G & \twoheadrightarrow & G'' & \twoheadrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \hookrightarrow & H' & \longrightarrow & H & \longrightarrow & H'' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0, & &
\end{array} \tag{5.3}$$

where the bottom row consists of the maximal affine k-subgroups of G', G and G'', the top row consists of the quotient Abelian k-varieties. As $H^d_{\{x\}}(X,\_)$ is exact on the columns of (5.2) (see Proposition 2.4.5) and the top row is mapped to 0 by Proposition 2.4.3, it is enough to show that the bottom row is mapped to a kernel. By taking homology of the rows of (5.2) we obtain the following long exact sequence:



As $H(G') \cong H(G) \cong H(G'') \cong 0$ we see that $H(H') \cong 0$ and $H(V') \cong H(H)$. As the k-group $H(V') \cong H(H)$ is both affine and projective it is finite étale (see Proposition 2.3.25) and as k is algebraically closed of characteristic 0 we see that it is constant (see Proposition 2.3.26). We thus see that H' is canonically isomorphic to the connected component of the kernel of $H \to H''$ and we se that

$$0 \to H^d_{\{x\}}(X, H') \to H^d_{\{x\}}(X, H) \to H^d_{\{x\}}(X, H'')$$

is exact in the same way as in the previous step.
Now we still have to show that $H^d_{\{x\}}(X, \_)$ takes short exact sequences of affine algebraic k-groups to kernels. Let

$$0 \to G' \to G \to G'' \to 0$$

be a short exact sequence of affine algebraic k-groups, then, because morphisms from k-groups of multiplicative type to unipotent k-groups and vice versa are zero, we see that the sequences

$$0 \to G'_m \to G_m \to G''_m \to 0 \qquad \text{and} \qquad 0 \to G'_u \to G_u \to G''_u \to 0$$

are exact. Now, both short exact sequences split; the first because the category of algebraic unipotent k-groups is equivalent to the category of finite k-vector spaces (see Proposition 2.3.48) and the second because it corresponds to a short exact sequence of finitely generated free Abelian groups (see Propositions 2.3.32 & 2.3.36), but such groups are projective so we are done. □

Notation 5.1.2. Let X be a connected regular k-scheme, then for every $i \geq 0$ we denote by $J_i(X)$ the pro-algebraic k-group representing the functor

$$\begin{array}{rcl} \mathbf{AGS}_k & \to & \mathbf{Ab} \\ G & \mapsto & \Gamma(X, C^i_{X,G}). \end{array}$$

⌟

Corollary 5.1.3. *The functor*

$$\begin{array}{rcl}\mathbf{AGS}_k & \to & \mathbf{Ch}_{\geq 0}(\mathbf{Ab}) \\ G & \mapsto & \Gamma(X, C^*_{X,G})\end{array}$$

is represented by a chain of pro-algebraic k-groups concentrated in non-negative degree.

Proof. For every $i \geq 0$ we obtain a diagram

$$\begin{array}{ccc}\mathbf{PAGS}_k(J_{i+1}(X), \_) & \xleftarrow{\cong} & \Gamma(X, C^{i+1}_{X,\_}) \\ & & \uparrow \\ \mathbf{PAGS}_k(J_i(X), \_) & \xleftarrow{\cong} & \Gamma(X, C^{i}_{X,\_}),\end{array}$$

in which the vertical morphism is a well defined natural transformation by Proposition 3.2.38. The maps $J_{i+1}(X) \to J_i(X)$ are thus obtained by the fact that the functor $\mathbf{PAGS}_k \hookrightarrow \widehat{\mathbf{AGS}_k}$ is fully faithful. It remains to show that the sequence

$$\cdots \to J_{i+1}(X) \to J_i(X) \to J_{i-1}(X) \to \cdots$$

is a chain complex, but the corresponding sequence of representable functors

$$\cdots \leftarrow \mathbf{PAGS}_k(J_{i+1}(X), \_) \leftarrow \mathbf{PAGS}_k(J_i(X), \_) \leftarrow \mathbf{PAGS}_k(J_{i-1}(X), \_) \leftarrow \cdots$$

is a cochain complex iff it is a cochain complex objectwise and this follows immediately from the fact that the sequence is isomorphic to

$$\cdots \leftarrow C^{i+1}_{X,\_} \leftarrow C^{i}_{X,\_} \leftarrow C^{i-1}_{X,\_} \leftarrow \cdots .$$

□

Definition 5.1.4. Let X be a connected regular k-scheme, then $J_*(X)$ is called the *pro-algebraic resolution* of X. ⌟

Corollary 5.1.5. *Let X be a connected regular k-scheme, then the cokernel of $J_1(X) \to J_0(X)$ represents the functor*

$$\begin{array}{rcl}\mathbf{AGS}_k & \to & \mathbf{Ab} \\ G & \mapsto & \mathbf{Sch}_k(X, G).\end{array}$$

Proof. Let G be an algebraic k-group, then, because representable Abelian presheaves take cokernels to kernels, we see that $\mathbf{PAGS}_k(\_, G)$ takes the cokernel $J_1(X) \to J_0(X) \to \mathrm{Cok}_{d_1}$ to $\mathbf{Sch}_k(X, G) \hookrightarrow \Gamma(X, C^0_{X,G}) \to \Gamma(X, C^1_{X,G})$. Functoriality follows from Proposition 3.2.38. □

Notation 5.1.6. Let X be a connected regular k-scheme, then we denote the cokernel of $J_1(X) \to J_0(X)$ by $J_{-1}(X)$. ⌟

Remark 5.1.7. Let X be a connected regular k-scheme. It is not necessary to first construct the resolution $J_*(X)$ in order to construct the pro-algebraic k-group $J_{-1}(X)$. The functor $G \mapsto \mathbf{Sch}_k(X, G)$ is left exact, so we may simply apply Corollary 1.1.11 to obtain $J_{-1}(X)$. ⌟

Remark 5.1.8. We are unaware whether or not the map $X \mapsto J_*(X)$ extends to a functor from connected regular k-schemes to cochains of pro-algebraic k-groups. What is certain is that such functoriality cannot be deduced from Cousin resolutions of sheaves associated to algebraic k-groups, because given an algebraic k-group G the map $X \mapsto C^*_{X,G}$ is not functorial; consider for example the case where $G = \mathbb{G}_a$ and in which we are given two integral k-algebras A, B as well as morphism $\varphi : \operatorname{Spec} A \to \operatorname{Spec} B$, then, if we write $X := \operatorname{Spec} A$ and $Y := \operatorname{Spec} B$, the square

$$\begin{array}{ccc} \Gamma(C^0_{X,G}, X) & \longrightarrow & \Gamma(C^0_{Y,G}, Y) \\ \uparrow & & \uparrow \\ \mathbf{Sch}_k(X, G) & \longrightarrow & \mathbf{Sch}_k(Y, G) \end{array}$$

would correspond to the square

$$\begin{array}{ccc} K(A) & \longleftarrow & K(B) \\ \uparrow & & \uparrow \\ A & \longleftarrow & B, \end{array}$$

which does not commute if $B \to A$ is not a monomorphism. ⌟

Remark 5.1.9. The map which assigns to any regular connected k-scheme the pro-algebraic group $J_{-1}(X)$ may however be extended to a functor due to the universal property of $J_{-1}(X)$. Note also that this universal property is similar to that of the Albanese variety of a pointed smooth projective variety over a perfect field (see [Moc12, Th. A.6]). ⌟

5.2 Properties of the pro-algebraic resolution

Theorem 5.2.1. *Let X be a connected regular k-scheme, then the pro-algebraic k-groups $J_i(X)$ are connected and affine for $i \geq 1$, and moreover unipotent for $i \geq 2$.*

Proof. By Theorem 1.4.13 we know that for each $i \geq 0$ the group $J_i(X)$ is the filtered limit of all its algebraic quotients.

We first show that for each $i \geq 1$ every algebraic quotient of $J_i(X)$ is connected. Let G be an algebraic quotient, then $\mathbf{PAGS}_k(J_i(X), \pi_0(G)) \cong \bigoplus H^i_{\{x\}}(\operatorname{Spec} \mathscr{O}_{X,x}, \pi_0(G)) \cong 0$, so that the composition of the morphisms $J_i(X) \twoheadrightarrow G \twoheadrightarrow \pi_0(G)$ and therefore $G \twoheadrightarrow \pi_0(G)$ is zero.

Similarly, we see that for each $i \geq 1$ the pro-algebraic k-group is affine, for if G is an algebraic quotient of $J_i(X)$ and V is the unique quotient of G which is an Abelian variety and such that the kernel of $G \twoheadrightarrow V$ is affine (see Theorem 2.3.24), then $\mathbf{PAGS}_k(J_i(X), V) \cong \bigoplus H^i_{\{x\}}(\operatorname{Spec} \mathscr{O}_{X,x}, V) \cong 0$, and just as in the previous paragraph we conclude that $V \cong 0$. The k-group $J_i(X)$ is then affine because any filtered limit (in $\mathbf{PAGS}_k$) of affine groups is affine (see [KS06, Prop. 6.1.9 & 6.1.10]).

We now also see that $J_i(X)$ is irreducible and thus connected for each $i \geq 1$ by the same argument as the one used in the proof of Lemma 2.3.15.
It remains to show that for $i \geq 2$ the groups $J_i(X)$ are unipotent. Let $i \geq 2$ and let G be an algebraic k-group. We have

$$\begin{aligned}\mathbf{PAGS}_k(J_i(X)_m, G_u) &\cong 0\\ \mathbf{PAGS}_k(J_i(X)_u, G_m) &\cong 0\end{aligned}$$

by Proposition 2.3.51, and

$$\mathbf{PAGS}_k(J_i(X), G_m) \cong \bigoplus_{x \in X^i} H^i_{\{x\}}(\operatorname{Spec} \mathscr{O}_{X,x}, G_m) \cong 0,$$

where the last isomorphism is due to the fact that any algebraic k-group of multiplicative type is the extension of a finite étale k-group by a group of the form $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$; for every $x \in X^i$ we thus obtain the exact sequence

$$\underbrace{H^i_{\{x\}}(\operatorname{Spec} \mathscr{O}_{X,x}, \mathbb{G}_m \times \cdots \mathbb{G}_m)}_{\cong 0} \to H^i_{\{x\}}(\operatorname{Spec} \mathscr{O}_{X,x}, G_m) \to \underbrace{H^i_{\{x\}}(\operatorname{Spec} \mathscr{O}_{X,x}, \pi_0(G_m))}_{\cong 0}$$

by Proposition 2.4.5, where the first term is isomorphic to 0 by Proposition 2.4.2 (and Corollary 2.3.3) and the second because $(\mathbb{G}_m)_{\operatorname{Spec} \mathscr{O}_{X,x}} \cong \mathscr{O}^\times_{X,x}$ has a flasque resolution of length 1 (see Example 3.2.33).
We then obtain the sequence of isomorphisms

$$\begin{aligned}\mathbf{PAGS}_k(J_i(X), G) &\cong \mathbf{PAGS}_k(J_i(X), G_m) \oplus \mathbf{PAGS}_k(J_i(X), G_u)\\ &\cong \mathbf{PAGS}_k(J_i(X), G_u)\\ &\cong \mathbf{PAGS}_k(J_i(X)_m, G_u) \oplus \mathbf{PAGS}_k(J_i(X)_u, G_u)\\ &\cong \mathbf{PAGS}_k(J_i(X)_u, G_u)\\ &\cong \mathbf{PAGS}_k(J_i(X)_u, G_u) \oplus \mathbf{PAGS}_k(J_i(X)_u, G_m)\\ &\cong \mathbf{PAGS}_k(J_i(X)_u, G)\end{aligned}$$

whose composition is easily seen to be natural in G. The projection $J_i(X) \to J_i(X)_u$ is then seen to be an isomorphism because the canonical functor $\mathbf{PAGS}_k \hookrightarrow \widehat{\mathbf{AGS}_k}$ is fully faithful.

□

There remains one property of pro-algebraic resolutions which we have not been able to prove. Grothendieck states that for any regular local ring $(A, \mathfrak{m})$ of dimension 1 we have $(J_{A/k})_m \cong \mathbb{G}_m$. Finally, we briefly discuss the question whether the objects of a pro-algebraic resolution are projective and whether they calculate the derived functor of $\mathbf{Sch}_k(X, \_) : \mathbf{AGS}_k \to \mathbf{Ab}$. We first note that by Proposition 1.4.6 the category $\mathbf{PAGS}_k$ is the opposite of a Grothendieck category, and thus has enough projective objects (see [KS06, Th. 9.6.2]); the derived functor of $\mathbf{Sch}_k(X, \_)$ therefore exists. By considering the short exact sequence $0 \to \mu_n \to G_m \xrightarrow{\cdot n} \mathbb{G}_m \to 0$ we may answer both questions raised above in the negative; in both situations the functor $\mathbf{Sch}_k(X, \_) \cong \mathbf{PAGS}_k(J_0(X), \_)$ would take the above short exact sequence to a short exact sequence; in the first case by assumption, and in the second case, because $H^1(X, \mu_n)$ vanishes as $(\mu_n)_X$ is flasque (see Proposition 2.4.2), but this is of course not true in general.

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