Differentiable sheaves III: Homotopical calculi and the smooth Oka principle

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Abstract

Let A and X be smooth manifolds with A closed, then it is a classical theorem that $C^\infty(A, X)$ carries the structure of a Fréchet manifold and that $C^\infty(A, X)$ has the "correct homotopy type". Traditionally, this can be made precise by saying that the underlying topological space of $C^\infty(A, X)$ has the same homotopy type as the set $C^0(A, X)$ equipped with the compact-open topology. In this article we prove a far-reaching generalisation of this statement: for A any nice, possibly infinite dimensional smooth manifold (e.g. a loop space), and X any object in $\mathbf{Diff}^\infty$ — the ∞-topos of homotopy-type-valued sheaves on the site of smooth manifolds — the shape of the mapping sheaf $\underline{\mathbf{Diff}}^\infty(A, X)$ is equivalent to the mapping-homotopy-type of the shapes of A and X. In a previous article we showed that the shape of any object in $\mathbf{Diff}^\infty$ coincides with classical notions of the object's underlying homotopy type, obtained e.g. using the smooth singular complex construction. To prove our main result we construct model structures as well as more general homotopical calculi on the ∞-category $\mathbf{Diff}^\infty$ (which restrict to its full subcategory of 0-truncated objects, $\mathbf{Diff}^\infty_{\leq 0}$) with shape equivalences as the weak equivalences. These tools are moreover developed in such a way so as to be highly customisable, with a view towards future applications e.g. in geometric topology.

Contents

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Introduction

This is the third an final instalment in a series of papers ([Clo24a] & [Clo24b]) which studies the ∞-topos $\mathbf{Diff}^r$ of differentiable sheaves, i.e., sheaves on the category of C^r-manifolds and C^r-maps ($0 \le r \le \infty$). In [Clo24b] we gave a new proof of the classical fact that the constant sheaf functor $\mathbf{Diff}^r \leftarrow \mathcal{S} : \pi^*$ admits a left adjoint $\pi_! : \mathbf{Diff}^r \to \mathcal{S}$ called the *shape* functor.

Let A be a CW-complex, and X an arbitrary topological space, then the internal mapping space $\underline{\mathbf{TSpc}}(A, X)$ (consisting of the set of continuous maps equipped with the compact-open topology) is a model for the mapping homotopy type of the homotopy types modelled by A and X. In the differentiable setting, when A and X are manifolds, with A closed, the set $\mathbf{Diff}^r(A, X)$ may be endowed with the structure of an infinite dimensional Fréchet manifold [GG73, Th. 1.11], and it is a folk theorem that its underlying homotopy type is again equivalent to $\mathcal{S}(\pi_! A, \pi_! X)$. By [Wal12, Lm A.1.7] the Fréchet manifold of smooth maps from A to X is canonically equivalent to the internal mapping sheaf $\underline{\mathbf{Diff}}^r(A, X)$. Moreover, the shape functor $\pi_! : \mathbf{Diff}^r \to \mathcal{S}$ commutes with products, so that we obtain a comparison morphism $\pi_! \underline{\mathbf{Diff}}^r(A, X) \to \mathcal{S}(\pi_! A, \pi_! X)$. A differentiable sheaf A is then said to satisfy the *differentiable Oka principle* if the map $\pi_! \underline{\mathbf{Diff}}^r(A, X) \to \mathcal{S}(\pi_! A, \pi_! X)$ is an isomorphism for all differentiable sheaves X (see [SS21]), and it is natural to ask for which differentiable sheaves the differentiable Oka principle holds. We obtain the following generalisation of the main statement of [BEBP19] (see Theorem 2.3.28).

Theorem A. *Any paracompact Hausdorff C^∞-manifold locally modelled on Hilbert spaces, nuclear Fréchet spaces, or nuclear Silva spaces satisfies the differentiable Oka principle.* □

In [Clo24b] we show that many ways of extracting homotopy types from differentiable sheaves, e.g., using a C^r-total singular complex construction, compute their shape. The theory in [Clo24b] relies crucially on the fact that the shape functor $\pi_! : \mathbf{Diff}^r \to \mathcal{S}$ is a left adjoint, and therefore preserves colimits.

The proof of Theorem A on the other hand boils down to showing that the shape functor $\pi_! : \mathbf{Diff}^r \to \mathcal{S}$ commutes with certain pullbacks — which is more difficult. Specifically, one needs a method for identifying morphisms $X \to Y$ in $\mathbf{Diff}^r$ such that any pullback along $X \to Y$ commutes with $\pi_!$. It turns out that for the ∞-toposes considered in this article it is possible to construct homotopical calculi (such as e.g. model structures) so that this is true whenever $X \to Y$ is a fibration (in the homotopical calculus). Thus, we are led to develop flexible tools for constructing such homotopical calculi, which we do using the theory of test categories. We discuss several further results beyond Theorem A in the following subsection.

Applications to geometric topology

Here we discuss some of the good properties of $\mathbf{Diff}^r_{\leq 0}$, the topos of set valued sheaves on manifolds, and illustrate how these might be relevant to problems in geometric topology, and in particular to Gromov's sheaf theoretic h-principle (these applications will not be further discussed in the body of this article; for more details see [Aya09], [RW11], [Dot14], [Kup19]).

Let $\mathbf{Emb}^\infty_d$ denote the topological category whose objects are the d-dimensional smooth manifolds, and where $\underline{\mathbf{Emb}}^\infty_d(M,N)$ is the set of smooth embeddings of M in N equipped with, equivalently, the underlying topology of the Fréchet manifold $\mathbf{Emb}^\infty_d(M,N)$ or the C^∞-compact-open topology. Recall that a sheaf F on $\mathbf{Emb}^\infty_d$ valued in topological spaces is ***invariant*** if the map $\underline{\mathbf{Emb}}^\infty_d(M,N) \times F(M) \to F(N)$ is continuous.

Fixing a smooth manifold N, the following are examples of invariant sheaves:

1. The sheaf $\underline{\mathrm{Imm}}(\_, N)$ sending each manifold M to the space of immersions of M in N.

2. The sheaf $\underline{\mathrm{Subm}}(\_, N)$ sending each manifold M to the space of submersion of M to N.

3. The sheaf Conf of configurations sending any manifold M to the space of finite subsets of M, topologised in such a way that points may "disappear off to infinity" when M is open (See [RW11, §3]).

An invariant sheaf F is ***microflexible*** ([RW11, Def. 5.1]) if for

(i) any polyhedron K,

(ii) any manifold M,

(iii) compact subsets $A \subseteq B \subseteq M$, and

(iv) subsets $U \subseteq V \subseteq M$ containing A and B, respectively,

the lifting problem

$$\begin{tikzcd} & \{0\} \times K \arrow[r] \arrow[dl] \arrow[d] & F(V) \arrow[d] \\ {[0,\varepsilon] \times K} \arrow[r] \arrow[urr, dashed] & {[0,1] \times K} \arrow[r] & F(U) \end{tikzcd} \tag{1}$$

admits a solution for some $0 < \varepsilon < 1$, possibly after passing to a smaller pair $U \subseteq V$ containing A and B, respectively. Examples 1. - 3. listed above are microflexible.

For any invariant sheaf F and any manifold M one may construct the *scanning map* (see [Fra11, Lect. 17])

$$\mathrm{scan}: F(M) \to \Gamma\big(\mathrm{Fr}(TM) \times_{\mathrm{O}_n} F(\mathbf{R}^n) \to M\big), \tag{2}$$

and F is said to *satisfy the h-principle on M* if the scanning map is an equivalence.

Theorem ([Fra11, Lect. 20])**.** *Every microflexible invariant sheaf satisfies the h-principle on any open manifold.* □

This is a very powerful theorem, as the study of $\Gamma\big(\mathrm{Fr}(TM) \times_{\mathrm{O}_n} F(\mathbf{R}^n) \to M\big)$ is often easier than that of $F(M)$.

Example. For $F = \underline{\mathrm{Imm}}(\_, N)$ (as in 1. above), the space $\Gamma\big(\mathrm{Fr}(TM) \times_{\mathrm{O}_n} F(\mathbf{R}^n) \to M\big)$ can with little effort be shown to be equivalent to the space of formal immersions of M into N, that is, the set of bundle maps

$$\begin{tikzcd} TM \arrow[r] \arrow[d] & TN \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

which restrict to monomorphisms $T_xM \to T_{fx}N$ for all $x \in M$. The h-principle can then e.g. be used to prove the famed Smale-Hirsch theorem (see [Sma59] & [Hir59] for details). ⌟

The above theorem may be viewed as a statement that any microflexible invariant sheaf $F : (\mathbf{Emb}_n^\infty)^{\mathrm{op}} \to \mathbf{TSpc}$ retains many of its exactness properties when composed with the functor $\mathbf{TSpc} \to \mathcal{S}$, sending any topological space to its (singular) homotopy type. The geometry of the constituent spaces of F is frequently crucial for proving microflexibility. However,

1. it is often difficult to construct suitable topologies on these spaces which exhibit this geometry, and
2. these topologies fail to account for natural smooth structures which one would expect these spaces to admit.

In fact, the constituent spaces of F are oftentimes more naturally viewed as objects of $\mathbf{Diff}^\infty$ (as already observed in [GTMW09] and [Kup19]), so that one is lead to consider sheaves of the form $F : (\mathbf{Emb}_n^\infty)^{\mathrm{op}} \to \mathbf{Diff}^\infty$. At a first glance, it may look as if we are introducing a new complication by considering sheaves valued in an ∞-category rather than an ordinary category. However, in most cases, such as in the examples 1. - 3. considered above, we obtain sheaves valued in $\mathbf{Diff}^\infty_{\leq 0}$. The following theorem provides a first justification for replacing $\mathbf{TSpc}$ with $\mathbf{Diff}^\infty_{\leq 0}$ (see Proposition 1.4.10 and Theorem 2.2.2).

Theorem B ([Cis03, §6.1])**.** *The topos $\mathbf{Diff}^\infty_{\leq 0}$ admits a model structure such that the restriction of the shape functor $\pi_! : \mathbf{Diff}^\infty_{\leq 0} \to \mathcal{S}$ exhibits $\mathcal{S}$ as a localisation of $\mathbf{Diff}^\infty_{\leq 0}$ along the weak equivalences.* □

Thus, many of the techniques developed in this article may be used without knowledge of ∞-categories. Moreover, $\mathbf{Diff}^\infty_{\leq 0}$ has excellent formal properties, which are directly relevant to the microflexibility condition ([Clo24a, Th. 2.16] & Corollary 1.1.5):

Theorem C. *Closed manifolds are categorically compact in $\mathbf{Diff}^\infty$ (and thus in $\mathbf{Diff}^\infty_{\leq 0}$).* □

Theorem D. *Filtered colimits in* $\mathbf{Diff}^{\infty}_{\leq 0}$ *are homotopy colimits.* □

To give a simple illustration of how these properties are relevant to the sheaf theoretic h-principle, we see that the lifting condition (1) may now be replaced with

$$\begin{tikzcd}
 & \{0\} \times K \arrow[dl] \arrow[d] \arrow[r] & \operatorname*{colim}_{V \supseteq B} F(V) \arrow[d] \\
{[0,\varepsilon] \times K} \arrow[r] \arrow[urr, dashed] & {[0,1] \times K} \arrow[r] & \operatorname*{colim}_{U \supseteq A} F(U)
\end{tikzcd} \tag{3}$$

eliminating the necessity to gradually choose smaller and smaller open neighbourhoods $V \supseteq U$ of $B \supseteq A$. Indeed, this is close to how Gromov originally formulated the microflexibility condition (see [Gro86, §1.4.2]) but instead using quasi-topological spaces (introduced by Spanier; [Spa63]) as a replacement for topological spaces, with the intention of obtaining well-behaved colimits (as explained in [Gro86, §1.4.1]). Unfortunately, both theorems D and C fail for quasi-topological spaces, as shown in the example below, so that (3) does not give the correct formulation of microflexbility in this setting.

A further use of the good formal properties of $\mathbf{Diff}^{\infty}_{\leq 0}$ is suggested by Ayala in [Aya09, p. 19]: A key step in the construction of the scanning map (2) involves carefully choosing a connection on M and then reparametrising the resulting exponential map $\exp : TM \to M$ (see [RW11, §6]). In order to formulate an h-principle which works for any exponential function, Ayala constructs the following variant of the scanning map given by

$$\mathrm{scan} : F(M) \to \Gamma\big(\mathrm{Fr}(TM) \times_{\mathrm{O}_n} \operatorname*{colim}_{\delta > 0} F\big(\mathring{B}^n{}_\delta(0)\big)\big). \tag{4}$$

The colimit $\operatorname{colim}_{\delta>0} F\big(\mathring{B}^n{}_\delta(0)\big)$ is again taken in the category of quasi-topological spaces in [Aya09] with the expectation that it has the same homotopy type as $F(\mathbf{R}^n)$, but this once more fails by the example below. Fortunately, by Theorem D the colimit does have the correct homotopy type when taken in $\mathbf{Diff}^{r}_{\leq 0}$. More generally, we believe that working with differentiable sheaves throughout in [Aya09] would fix issues which arise from working with quasi-topological spaces.

Example. For each $\delta > 0$ the space $\mathrm{Conf}\big(\mathring{B}^n_\delta(0)\big)$ is weakly equivalent to S^n by the subsequent theorem. In Ayala's variant of quasi-topological spaces (see [Aya09, Def. 2.7]) the colimit is equivalent to the Sierpinski space, which is contractible. In other variants of quasi-topological spaces (e.g., [SW57, §3], [Gro86, §1.4.1]) one still obtains a contractible two-point space. ⌟

Configuration spaces We conclude this subsection with a proof of the following fact using the techniques introduced here.

Theorem. *The space* $\mathrm{Conf}(\mathbf{R}^n)$ *is weakly equivalent to* S^n *for any* $n \geq 0$.

For any smooth manifold M we first redefine $\mathrm{Conf}(M)$ to be the differentiable sheaf which associates to any Cartesian space $\mathbf{R}^d$ the set of embeddings $C \hookrightarrow M \times \mathbf{R}^d$ such that the map $C \to M$ is a submersion with 0-dimensional fibres. Using the smoothing argument in [GRW10, Lm. 2.17] one can show that the singular homotopy type of $\mathrm{Conf}(M)$ as a topological space coincides with its shape as a differentiable sheaf. (Note that the definition of $\mathrm{Conf}(M)$ as a differential sheaf is much simpler than the definition of $\mathrm{Conf}(M)$ as a topological space.)

We can now prove the theorem based on an idea originally due to Segal ([Seg79, Prop. 3.1]):

Sketch of proof. For every $\varepsilon > 0$ denote by $\mathrm{Conf}_\varepsilon(\mathbf{R}^n)$ (resp. $\mathrm{Conf}_{\leq 1}(\mathbf{R}^n)$) the subspace of $\mathrm{Conf}(\mathbf{R}^n)$ consisting of those configurations containing at most one point in $\mathring{B}_\varepsilon(0)$ (resp., all of $\mathbf{R}^n$), then $\mathrm{Conf}_{\leq 1}(\mathbf{R}^n)$ may be exhibited as a retract of $\mathrm{Conf}_\varepsilon(\mathbf{R}^n)$ by pushing all points outside of $\mathring{B}_\varepsilon(0)$ in any configuration in $\mathrm{Conf}_\varepsilon(\mathbf{R}^n)$ off to infinity. Moreover, $\mathrm{Conf}_{\leq 1}(\mathbf{R}^n)$ is $\mathbf{R}$-homotopy equivalent to S^n, as $\mathrm{Conf}_{\leq 1}(\mathbf{R}^n)$ is essentially the one-point-compactification of $\mathbf{R}^n$. Finally, we have $\mathrm{colim}_{\varepsilon > 0} \mathrm{Conf}_\varepsilon(\mathbf{R}^n) = \mathrm{Conf}(\mathbf{R}^n)$, so that

$$\pi_! \,\mathrm{Conf}(\mathbf{R}^n) = \pi_! \operatorname*{colim}_{\varepsilon>0} \mathrm{Conf}_\varepsilon(\mathbf{R}^n) = \operatorname*{colim}_{\varepsilon>0} \pi_! \mathrm{Conf}_\varepsilon(\mathbf{R}^n) = \operatorname*{colim}_{\varepsilon>0} \pi_! S^n = \pi_! S^n,$$

where the second equivalence follows from Theorem D. □

Organisation

This is the third and final article of a trilogy. Like [Clo24a] and [Clo24b], the present article is split into two parts where the first part develops a piece of toposic technology, in this case homotopical calculi in on locally contractible ∞-toposes, and then applies it to the ∞-topos $\mathbf{Diff}^r$.

1 Homotopy theory in locally contractible (∞-)toposes: Given an ∞-category C together with a subcategory W of weak equivalences, we discuss which (co)limits in C are preserved by the localisation functor $\gamma : C \to W^{-1}C$, i.e, which (co)limits are *homotopy (co)limits*. In all of our applications C will be a subcategory of an ∞-topos $\mathcal{E}$ and γ, the restriction of the shape functor $\pi_! : \mathcal{E} \to \mathcal{S}$ to C, so the relationship between homotopy limits and colimits is not symmetric: the functor $\pi_! : \mathcal{E} \to \mathcal{S}$ preserves *all* colimits but only certain limits. Thus, any colimit in C which commutes with the inclusion $C \hookrightarrow \mathcal{E}$ is a homotopy colimit. In §1.1 we study which colimits are preserved by $C \hookrightarrow \mathcal{E}$ when C is the subcategory of n-truncated objects for some $0 \leq n \leq \infty$, and in §1.2 we study the situation when $\mathcal{E}_{\leq 0}$ is a local topos, and C is the ordinary category of concrete 0-truncated sheaves on $\mathcal{E}$. In §1.3 we discuss how to recognise homotopy limits for arbitrary localisations using homotopical calculi, such as model structures (on ∞-categories). In particular, we show how to recognise *sharp morphisms* — morphisms along which all pullbacks are homotopy pullbacks. Finally, in §1.4 we combine the theory of test categories with the technology of [Clo24b, §1.2.1] to construct model structures on locally contractible ∞-toposes as well as ordinary toposes generated by objects of contractible shape.

2 Homotopical calculi on differentiable sheaves: We use the results of §1 to study $\mathbf{Diff}^r$ and to prove Theorem A. In more detail: First we recall some basic facts about diffeological spaces in §2.1. Then, in §2.2 we show that $\mathbf{Diff}^r_{\leq 0}$ and the subcategory of diffeological spaces, $\mathbf{Diff}^r_{\mathrm{concr}}$, model $\mathcal{S}$. Moreover, we construct numerous model structures on $\mathbf{Diff}^r$, $\mathbf{Diff}^r_{\leq 0}$, and $\mathbf{Diff}^r_{\mathrm{concr}}$ (in which the weak equivalences are the shape equivalences). With a bit of extra work, we also show that it is possible to recover the Quillen equivalence $\widehat{\Delta} \rightleftarrows \mathbf{TSpc}$. Finaly, §2.3 is devoted to the proof of Theorem A: The main idea is to show that the class of objects satisfying the differentiable Oka principle is closed under various (co)limits and under Δ^1-homotopy equivalence. Then, one may show inductively that simplicial complexes built using Kihara's simplices (see [Kih19, § 1.2] or [Clo24b, Def. 2.12]) satisfy the differentiable Oka principle, and that the manifolds in Theorem A are Δ^1-homotopy equivalent to such simplicial complexes. The above induction step relies on showing that for each differentiable sheaf X and each Kihara boundary inclusion $\partial\Delta^n \hookrightarrow \Delta^n$ the map $\underline{\mathbf{Diff}}^\infty(\partial\Delta^n, X) \leftarrow \underline{\mathbf{Diff}}^\infty(\Delta^n, X)$ is sharp, which we do by exhibiting it as a *squishy fibration*; a notion which we introduce in §2.3 for precisely this purpose. We conclude §2.3 by providing examples of manifolds which do *not* satisfy the differentiable Oka principle.

The article includes three appendices collecting some necessary background material:

§A recalls some basic facts about the cube category and cubical diagrams.

§B exhibits how the definition of model structures may be implemented in the ∞-categorical setting.

§C provides some (mostly new) results on pro-objects in ∞-categories (which are however already well-known in the ordinary categorical setting).

Relation to other work

A model structure on $\mathbf{Diff}^r_{\leq 0}$ in which the weak equivalence are given by the shape equivalences is first provided in [Cis03, §6.1]. We should like to point out that many results in this article (in particular on locally contractible ∞-toposes and cofinality) are ultimately the product of us trying to understand [Cis03] and [Cis06] in ∞-categorical terms. A different construction of one of the model structures described in 2.2.5 is given in [Pav22, Th. 7.4].

Theorem A is a generalisation of the main theorem of [BEBP19], but relies on a careful analysis of the shape functor and its relationship to homotopical calculi rather than the combinatorics of simplicial sets. An important inspiration for adopting a more flexible attitude towards homotopical calculi is given in [Cis19, §7], and we should like to point out that a proof of Theorem A in the vein of this article would be significantly harder without Kihara's simplices (see [Kih20]).

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1 Homotopy theory in locally contractible (∞-)toposes

Fix a ***relative ∞-category*** (C, W), i.e. an ∞-category C together with a subcategory W (whose morphisms are called ***weak equivalences***) containing all isomorphisms. It is then natural to study the relationship between C and its localisation $W^{-1}C$; in particular, one may ask which limits in $W^{-1}C$ may be obtained via constructions in C.

Definition 1.0.1. Let K be a simplicial set, then a functor $p : K^{\triangleleft} \to C$ is called a ***homotopy limit*** of $p|_K : K \to C$ if the composition of $K^{\triangleleft} \to C \to W^{-1}C$ is a limit of the composition of $K \xrightarrow{p|_K} C \to W^{-1}C$. A functor $K^{\triangleright} \to C$ is a ***homotopy colimit*** if $(K^{\triangleright})^{\mathrm{op}} \to C^{\mathrm{op}}$ is a homotopy limit. ⌟

In particular, a (co)limit in C is a homotopy (co)limit iff it is carried to a (co)limit by $C \to W^{-1}C$.

Recall that an ∞-topos $\mathcal{E}$ is *locally contractible* if the unique left exact cocontinuous functor $\mathcal{E} \leftarrow \mathcal{S} : \pi^*$ admits a left adjoint, $\pi_! : \mathcal{E} \to \mathcal{S}$, and that the induced functor $(\pi_!)_{/\mathbf{1}_{\mathcal{E}}} : \mathcal{E} \to \mathcal{S}_{/\pi_!\mathbf{1}_{\mathcal{E}}}$ is a localisation (see [Clo24b, §1.2]). While at the level of generality of Definition 1.0.1 the theories of homotopy limits and colimits are dual to each other, in this article homotopy limits and colimits have very different flavours. This is because the localisation functors under consideration of are all of the form $C \to \mathcal{S}$ with C some subcategory of $\mathcal{E}$, and the localisation functor is simply given by the restriction of $\pi_!$ to C. Thus, when $C = \mathcal{E}$ all colimits are homotopy colimits. When $C \subsetneq \mathcal{E}$ we can exhibit many colimits in C as homotopy colimits by showing that they are preserved by the inclusion $C \hookrightarrow \mathcal{E}$. This approach is explored in §1.1 & §1.2 where C consists of n-truncated objects and concrete objects (to which we also give a brief introduction) respectively.

Commuting limits past $(\pi_!)|_C$ is considerably harder and requires different techniques. To this end we develop the basic theory of homotopical calculi (e.g. model structures) on ∞-categories in §1.3, and then use the machinery developed in [Clo24b, §1.2.1] combined with test categories to construct homotopical calculi on locally contractible ∞-toposes in §1.4.

1.1 Colimits of n-truncated objects in ∞-toposes

Let $\mathcal{E}$ be a fixed ∞-topos, and $n \geq -2$. In this subsection we will show that many colimits of n-truncated objects in $\mathcal{E}$ are again n-truncated.

Proposition 1.1.1. *Consider a pushout square in* $\mathcal{E}$

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, hook] & Y' \end{tikzcd}$$

for which X, X', Y are n-truncated and in which the top horizontal map (and thus also the bottom horizontal map; see [ABFJ20, Prop. 2.2.6]) is a monomorphism, then Y' is n-truncated. □

Proposition 1.1.2. *The inclusion $\mathcal{E}_{\leq n} \hookrightarrow \mathcal{E}$ commutes with filtered colimits.* □

Proposition 1.1.3. *The inclusion $\mathcal{E}_{\leq n} \hookrightarrow \mathcal{E}$ commutes with coproducts.* □

Proposition 1.1.4. *The subcategory of n-truncated objects is closed under retracts.* □

Corollary 1.1.5. *Let A be a small category, and $X : A \to \mathcal{E}_{\leq n}$ a functor. If either*

1. *X is a wedge in which one leg is a monomorphism,*
2. *A is filtered, or*
3. *A is discrete,*

then the restricted shape functor $\pi_!|_{\mathcal{E}_{\leq n}} \to \mathrm{Pro}(\mathcal{S})$ preserves the colimit of X . □

Discussion of the proofs of Propositions 1.1.1 - 1.1.4 All four propositions may be proved by first checking the statement for simplicial sets equipped with the Kan-Quillen model structure, so that they are true in $\mathcal{S}$. In any presheaf ∞-topos the statements can be checked pointwise. The general statements then follow from the fact that left exact functors preserve monomorphisms and truncation.

We would find it conceptually pleasing to have proofs of these statements which rely on descent (similar to e.g. [ABFJ20, Prop. 2.2.6]) rather than the fact that every ∞-topos is a left exact localisation of a presheaf ∞-category. A proof of a generalisation of Proposition 1.1.4 in this style was suggested to us by Bastiaan Cnossen.

Proposition 1.1.6. *Let C be a finitely complete ∞-category, then n-truncated maps in C are closed under retracts.*

Proof. Let

$$\begin{tikzcd} x' \arrow[r] \arrow[d] & x \arrow[r] \arrow[d] & x' \arrow[d] \\ y' \arrow[r] & y \arrow[r] & y' \end{tikzcd}$$

be a retract diagram in which $x \to y$ is n-truncated, then we wish to show that $x' \to y'$ is likewise n-truncated. For $n = -2$ the statement is clear, so assume that $n > -2$. Then we obtain a new retract diagram

$$\begin{tikzcd} x' \arrow[r] \arrow[d] & x \arrow[r] \arrow[d] & x' \arrow[d] \\ x' \times_{y'} x' \arrow[r] & x \times_y x \arrow[r] & x' \times_{y'} x' \end{tikzcd}$$

and the general statement follows by induction. $\square$

1.2 Concrete objects

Throughout this subsection $\mathcal{E}$ denotes an *ordinary* topos. Recall that $\mathcal{E}$ is *local* if the global sections functor $\pi_* : \mathcal{E} \to \mathbf{Set}$ admits a right adjoint $\mathcal{E} \leftarrow \mathbf{Set} : \pi^!$, which by the same argument as for ∞-toposes is fully faithful. We first define the full subcategory $\mathcal{E}_{\text{concr}}$ of concrete objects in a local topos $\mathcal{E}$ and discuss some of its basic properties before exhibiting various colimits which are preserved by the inclusion $\mathcal{E}_{\text{concr}} \hookrightarrow \mathcal{E}$ in §1.2.1.

Definition 1.2.1. An object X in $\mathcal{E}$ is ***concrete*** if the canonical morphism $X \to \pi^!\pi_*X$ is a monomorphism. The subcategory of $\mathcal{E}$ spanned by concrete objects is denoted by $\mathcal{E}_{\text{concr}}$. ⌟

A concrete object in $\mathcal{E}$ may be thought of a set together with extra structure, making it into an object in $\mathcal{E}$. The functor $\pi_* : \mathcal{E}_{\text{concr}} \to \mathbf{Set}$ is moreover faithful (but not full, in general). To see this let X, Y be two concrete objects together with morphisms $X \rightrightarrows Y$ whose image agree in $\mathbf{Set}(\pi_*X, \pi_*Y)$, then we obtain a diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, shift left] \arrow[d, shift right] & \pi^!\pi_*X \arrow[d] \\ Y \arrow[r, hook] & \pi^!\pi_*Y, \end{tikzcd}$$

and we see that $X \rightrightarrows Y$ are equalised by the monomorphism $Y \hookrightarrow \pi^!\pi_*Y$. Thus a morphism $X \to Y$ in $\mathcal{E}_{\mathrm{concr}}$ may be viewed as a morphism $\pi_*X \to \pi_*Y$ on underlying sets, respecting the extra structure making the sets π_*X, π_*Y into objects in $\mathcal{E}_{\mathrm{concr}}$. This perspective is used for instance in Example 1.2.7.

Example 1.2.2. For any small category A which admits a final object, the topos $\widehat{A}$ is local. To see this, observe that $\pi_* : \widehat{A} \to \mathbf{Set}$ is simply given by evaluation at the final object, and thus commutes with colimits; therefore, it admits a right adjoint by the adjoint functor theorem, which is given by sending any set X to $a \mapsto \mathbf{Set}(A(\mathbf{1}_A, a), X)$. Concrete objects in $\widehat{A}$ are then referred to as ***concrete presheaves on*** A. A concrete presheaf on A is given by a set X together with a subset of $\mathbf{Set}(A(\mathbf{1}_A, a), X)$ for every object a in A; these subsets are then required to be closed under precomposing by morphisms in A.

This observation applies to the topos of simplicial sets $\widehat{\Delta}$, where the functor $\pi^!$ is exhibited by $\mathrm{cosk}_0 : \mathbf{Set} \hookrightarrow \widehat{\Delta}$. The concrete objects are then those simplicial sets X such that for any $(n+1)$-tuple $(x_0, \ldots, x_n) \in X_0^{(n+1)}$ there exists at most one n-simplex with precisely these vertices. ⌟

Proposition 1.2.3. *The inclusion $\mathcal{E}_{\mathrm{concr}} \hookrightarrow \mathcal{E}$ admits a left adjoint.*

Proof. Recall that in any topos the epimorphisms and the monomorphisms form an orthogonal factorisation system. Let X be an object in $\mathcal{E}$, then $X \to \pi^!\pi_*X$ may be factored uniquely as $X \twoheadrightarrow X' \hookrightarrow \pi^!\pi_*X$. Consider any map $X \to Y$, where Y is concerete, then the lifting problem

$$\begin{tikzcd} X \arrow[rr] \arrow[d, two heads] & & Y \arrow[d, hook] \\ X' \arrow[r, hook] & \pi^!\pi_*X \arrow[r] & \pi^!\pi_*Y \end{tikzcd}$$

admits a unique solution, exhibiting the universality of $X \to X'$. □

Definition 1.2.4. The left adjoint of the inclusion $\mathcal{E}_{\mathrm{concr}} \hookrightarrow \mathcal{E}$ (which exists by the preceding proposition) is called the ***concretisation***. ⌟

Proposition 1.2.5. *The category $\mathcal{E}_{\mathrm{concr}}$ is presentable.*

Proof. The pair $(\pi^!, \pi_*)$ is a geometric embedding, so that $\mathbf{Set}$ is a κ-accessible subcategory of $\mathcal{E}$ for some regular cardinal κ, i.e. $\pi^! : \mathbf{Set} \hookrightarrow \mathcal{E}$ commutes with κ-filtered colimits. We claim that $\mathcal{E}_{\mathrm{concr}} \hookrightarrow \mathcal{E}$ likewise commutes with κ-filtered colimits. Let A be a κ-filtered category, and consider a functor $X : A \to \mathcal{E}_{\mathrm{concr}}$; as filtered colimits, and a fortiori κ-filtered colimits preserve monomorphisms, the canonical map $\operatorname{colim} X \to \operatorname{colim} \pi^!\pi_*X \xrightarrow{\cong} \pi^!\pi_* \operatorname{colim} X$ is a monomorphism, so that $\operatorname{colim} X$ is concrete. □

1.2.1 Colimits of concrete objects in a local topos

Let $\mathcal{F}$ be an ∞-topos for which $\mathcal{F}_{\leq 0} = \mathcal{E}$. We now discuss which colimits in $\mathcal{E}_{\mathrm{concr}}$ are preserved by the inclusion $\mathcal{E}_{\mathrm{concr}} \hookrightarrow \mathcal{F}$.

Definition 1.2.6. A monomorphism $X \hookrightarrow Y$ in $\mathcal{E}_{\text{concr}}$ is called an ***embedding*** if

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, hook] & Y \arrow[d, hook] \\
\pi^! \pi_* X \arrow[r, hook] & \pi^! \pi_* Y
\end{tikzcd}$$

is a pullback square. ⌟

Example 1.2.7. Any retract $X \overset{i}{\hookrightarrow} Y \overset{r}{\twoheadrightarrow} X$ is an embedding. To see this, for any object Z and any morphisms f, g making the outer square in the diagram

$$\begin{tikzcd}
Z \arrow[dr, dashed, "h"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& X \arrow[r, hook] \arrow[d, hook, "\iota_X"'] & Y \arrow[l, bend right, two heads] \arrow[d, hook, "\iota_Y"] \\
& \pi^! \pi_* X \arrow[r, hook] & \pi^! \pi_* Y \arrow[l, bend right, two heads]
\end{tikzcd}$$

commute, there exists a unique map $h : \pi_* Z \to \pi_* X$ (indicated by the dashed arrow in the diagram), such that the triangles (ι_X, f, h) and (i, g, h) commute on *underlying sets*. We must show that h is in the image of $\mathcal{E}_{\text{concr}}(Z, Y) \hookrightarrow \mathbf{Set}(\pi_* Z, \pi_* X)$. Indeed, h may be written as $\pi_* i \circ \pi_* r \circ h$, and $\pi_* r \circ h = \pi_* g$ by assumption, so that $h = \pi_*(r \circ g)$. ⌟

Proposition 1.2.8. *Consider a span $Z \hookleftarrow X \hookrightarrow Y$ in $\mathcal{E}_{\text{concr}}$, where $X \hookrightarrow Y$ is monomorphism, and $X \hookrightarrow Z$ is an embedding, then the pushout of the above diagram in $\mathcal{F}$ is again an object of $\mathcal{E}_{\text{concr}}$.*

Proof. By Proposition 1.1.1 it is enough to show that the pushout in $\mathcal{E}$ is again in $\mathcal{E}_{\text{concr}}$.

We must show that the map $Y \sqcup_X Z \to \pi_! \pi_* (Y \sqcup_X Z)$ in

$$\begin{tikzcd}[row sep=small, column sep=small]
& Z \arrow[rr, hook] \arrow[dd, hook] & & Y \sqcup_X Z \arrow[dd] \\
X \arrow[ur, hook] \arrow[rr, hook, crossing over] \arrow[dd, hook] & & Y \arrow[ur, hook] & \\
& \pi^! \pi_* Z \arrow[rr, hook] & & \pi^! \pi_* (Y \sqcup_X Z) \\
\pi^! \pi_* X \arrow[ur, hook] \arrow[rr, hook] & & \pi^! \pi_* Y \arrow[ur, hook] \arrow[from=uu, hook, crossing over] &
\end{tikzcd} \tag{5}$$

is a monomorphism. Let T be any object in $\mathcal{E}$, and $f, g : T \rightrightarrows Y \sqcup_X Z$, any pair of morphisms, then we will show that if their compositions with $Y \sqcup_X Z \to \pi_! \pi_* (Y \sqcup_X Z)$ are equal, then so are f and g.

First, we consider the *special case* in which each of the morphisms f and g factor through either $Y \hookrightarrow Y \sqcup_X Z$ or $Z \hookrightarrow Y \sqcup_X Z$. If both f and g together factor through the same inclusion, then $f = g$

because $Y \hookrightarrow \pi^! \pi_*(Y \sqcup_X Z)$ and $Z \hookrightarrow \pi^! \pi_*(Y \sqcup_X Z)$ are monomorphisms. Thus, assume w.l.o.g. that f factors through Y and g factors through Z. Observe that the bottom square in (5) is a pullback square by Proposition 1.1.1, [ABFJ20, Prop. 2.2.6], and the fact that π_* and $\pi^!$ preserve limits, so that we obtain a morphism $T \to \pi^! \pi_* X$ from the commutative square

$$\begin{tikzcd} & \pi^!\pi_* Z \arrow[dr, hook] & \\ T \arrow[ur] \arrow[dr] & & \pi^!\pi_*(Y \sqcup_X Z) \\ & \pi^!\pi_* Y \arrow[ur, hook] & \end{tikzcd}$$

and thus a morphism $T \to X$ from the induced commutative square

$$\begin{tikzcd} T \arrow[r] \arrow[d] & Z \arrow[d, hook] \\ \pi^!\pi_* X \arrow[r, hook] & \pi^!\pi_* Z \end{tikzcd}$$

and the fact that $X \hookrightarrow Z$ is an embedding. The composition of $T \to X \to Z$ yields g by construction. To see that the composition of $T \to X \to Y$ yields f we further compose with the monomorphism $Y \hookrightarrow \pi^! \pi_*(Y \sqcup_X Z)$ which is equal to f composed with the same monomorphism.

For the general statement consider the effective epimorphism $\bigcup_{i=1}^4 U_i \to T$, where

$$\begin{aligned} U_1 &= f^*Y \times_{Y \sqcup_X Z,\, g|_{f^*Y}} Y \\ U_2 &= f^*Y \times_{Y \sqcup_X Z,\, g|_{f^*Y}} Z \\ U_3 &= f^*Z \times_{Y \sqcup_X Z,\, g|_{f^*Z}} Y \\ U_4 &= f^*Z \times_{Y \sqcup_X Z,\, g|_{f^*Z}} Z \end{aligned}$$

then the compositions of $U_i \to T \xrightarrow{f} Y \sqcup_X Z$ and $U_i \to T \xrightarrow{g} Y \sqcup_X Z$ factor through Y or Z for all i. By the above discussion, $\bigcup_{i=1}^4 U_i \to T$ equalises $T \rightrightarrows Y \sqcup_X Z$, and thus $f = g$, as $\bigcup_{i=1}^4 U_i \to T$ is an effective epimorphism. □

Lemma 1.2.9. *Let $\mathcal{G}$ be an ∞-topos, I a filtered category, and $X : I \to \mathcal{G}$ a diagram such that $X_i \hookrightarrow X_j$ is a monomorphism for all morphisms $i \to j$ in I, then $X_i \to \operatorname{colim} X$ is likewise a monomorphism for all i in I.*

Proof. Denote by $I_{i\leq}$ the full subcategory of I spanned by those objects admitting a morphism from i, then $I_{\leq i}$ is again filtered, and the functor $I_{i\leq} \to I$ is final, so that the canonical morphism $\operatorname{colim}_{k \in I_{i\leq}} X_k \to \operatorname{colim}_{k\in I} X_k$ is an isomorphism. As $I_{i\leq}$ is filtered, and thus connected, the morphism $X_i \hookrightarrow \operatorname{colim}_{k\in I_{i\leq}} X_k$ may be written as $\operatorname{colim}_{k\in I_{i\leq}} X_i \to \operatorname{colim}_{k \in I_{i\leq}} X_k$, and is a monomorphism, because filtered colimits commute with finite limits in ∞-toposes. □

Proposition 1.2.10. *Let I be a filtered category, and $X : I \to \mathcal{E}_{\mathrm{concr}}$ a diagram such that $X_i \to X_j$ is a monomorphism for all morphisms $i \to j$ in I, then the colimit of X in $\mathcal{F}$ is again in $\mathcal{E}_{\mathrm{concr}}$.*

Proof. By Proposition 1.1.2 it is enough to show that the colimit of $X : I \to \mathcal{E}$ in $\mathcal{E}_{\mathrm{concr}}$ is again in $\mathcal{E}_{\mathrm{concr}}$.

Denote by X the colimit of $X : I \to \mathcal{E}$. Let T be any object in $\mathcal{E}$, and $f, g : T \rightrightarrows X$ be a pair of morphisms, then we will show that if their compositions with $X \to \pi^! \pi_* X$ are equal, then so are f and g. By the same technique used in the last paragraph of the proof of Proposition 1.2.8 we may assume that f and g each factor through $X_i \to X$ and $X_j \to X$ respectively, and by the filteredness of I we may assume w.l.o.g. that $i = j$. Consider the square

$$\begin{tikzcd} X_i \arrow[r, hook] \arrow[d, hook] & X \arrow[d] \\ \pi^! \pi_* X_i \arrow[r, hook] & \pi^! \pi_* X \end{tikzcd}$$

in which $X_i \hookrightarrow X$ is a monomorphism by Lemma 1.2.9, and therefore also $\pi^! \pi_* X_i \hookrightarrow \pi^! \pi_* X$, as $\pi^! \pi_*$ preserves limits. The compositions of the lifts of f and g to $T \to X_i$ with the monomorphism $X_i \hookrightarrow \pi^! \pi_* X$ are equal by assumption, and thus so are f and g. $\square$

Proposition 1.2.11. *Any coproduct of concrete objects in $\mathcal{F}$ is again in $\mathcal{E}_{\text{concr}}$.*

Proof. By Proposition 1.1.3 it is enough to show that any coproduct of concrete objects in $\mathcal{E}$ is again in $\mathcal{E}_{\text{concr}}$.

Claim: For any object E in $\mathcal{E}$ the map $\varnothing \to E$ is an embedding.

By induction it then follows form Proposition 1.2.8 that any finite coproduct of concrete objects is concrete. An arbitrary coproduct is the filtered colimit of all its finite subcoproducts so that the proposition follows from Proposition 1.2.10.

Proof of claim: We must show that

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \pi^! \pi_* \varnothing \arrow[d] \\ E \arrow[r] & \pi^! \pi_* E \end{tikzcd}$$

is a pullback. The claim will follow from showing that for any map $A \to \pi^! \pi_* \varnothing$ we must have $A = \varnothing$. As π_* is a left adjoint we have $\pi_* \varnothing = \varnothing$, so that $A \to \pi^! \pi_* \varnothing = \pi^! \varnothing$ corresponds to a map $\pi_* A \to \varnothing$ so that $\pi_* A = \varnothing$. But then we have $A \to \pi^* \pi_* A = \varnothing$, so that $A = \varnothing$. $\square$

We then obtain the following corollary of the above propositions:

Corollary 1.2.12. *Let A be a small category, and $X : A \to \mathcal{E}_{\text{concr}}$ a functor. If either*

1. *X is a wedge in which one leg is an embedding, and the other a monomorphism,*

2. *A is filtered, or*

3. *A is discrete,*

then restricted shape functor $\pi_!|_{\mathcal{E}_{\text{concr}}} \to \mathrm{Pro}(\mathcal{S})$ preserves the colimit of X . $\square$

1.3 Basic theory of homotopical calculi

Here we construct homotopy (co)limits in a general relative ∞-category (C, W). Let us begin with the simplest case of a homotopy (co)limit: by [Cis19, Prop. 7.1.10] the localisation functor $\gamma : C \to W^{-1}C$ is both initial and final, so that if x_0 is an initial or final object of C, then $\gamma(x_0)$ is an initial or final object of $W^{-1}C$. Thus, if C has a final object, then $W^{-1}C$ admits all finite limits iff it admits all pullbacks, and admits all limits if it furthermore admits all products. Thus, we will focus on the construction of homotopy pullbacks. This leads us to consider the following definition.

Definition 1.3.1. A morphism $x' \to x$ in C is ***sharp*** if for every morphism $y \to x$ the pullback along $x' \to x$ exists and is a homotopy pullback (see Remark 1.3.5). ⌟

In order to recognise sharp morphisms, we abstract the properties of right proper model categories.

Definition 1.3.2. An object x in C is called ***right proper*** if the canonical functor

$$W_{/x}^{-1}C_{/x} \to (W^{-1}C)_{/x}$$

is an equivalence. The relative category (C, W) is ***right proper*** if all objects in C are right proper. ⌟

Notation 1.3.3. If an object x in C is right proper, then we will denote the ∞-category $(W^{-1}C)_{/x}$ by $W^{-1}C_{/x}$. ⌟

Remark 1.3.4. A model category is right proper in the usual sense iff its underlying relative category is right proper. This may be seen by combining [Rez98, Prop. 2.7] with [Cis19, Cor. 7.6.13][1]. ⌟

Remark 1.3.5. Let $f : x' \to x$ be a morphism in C, then recall that it is *sharp* in the sense of Rezk (see [Rez98, §2]), if for every morphism $b \to x$ and every weak equivalence $a \xrightarrow{\sim} b$ there exists a diagram

$$\begin{tikzcd} a' \arrow[r] \arrow[d] & b' \arrow[r] \arrow[d] & x' \arrow[d] \\ a \arrow[r, "\sim"] & b \arrow[r] & x \end{tikzcd} \tag{6}$$

in which all squares are pullbacks and such that $a' \to b'$ is a weak equivalence. If (C, W) is right proper, then a morphism in C is sharp in our sense iff it is sharp in the sense of Rezk.

To see this, first assume that $x' \to x$ is sharp in our sense, then it is sharp in the sense of Rezk, because for every diagram of the form (6) the rightmost and outer squares are homotopy pullbacks, so that the leftmost square is a homotopy pullback. Thus, if $a \to b$ is a weak equivalence, then $a' \to b'$ is a weak equivalence.

Conversely, if $x' \to x$ is sharp in the sense of Rezk, then the functor $C_{/x'} \leftarrow C_{/x} : f^*$ preserves weak equivalences, so that [Cis19, Prop. 7.1.14] yields, canonically, a commutative diagram

$$\begin{tikzcd} C_{/x'} \arrow[r, shift left, "f_!"] \arrow[r, phantom, "\perp"] \arrow[d] & C_{/x} \arrow[l, shift left, "f^*"] \arrow[d] \\ W^{-1}C_{/x'} \arrow[r, shift left, "f_!"] \arrow[r, phantom, "\perp"] & W^{-1}C_{/x} \arrow[l, shift left, "f^*"] \end{tikzcd}$$

[1] Rezk's proof of [Rez98, Prop. 2.7] can be interpreted verbatim in model ∞-categories, so that the remark is in fact true for model ∞-categories.

The pullback of any morphism $y \to x$ along f in C thus yields the pullback of $y \to x$ along f in $W^{-1}C$. ⌟

Luckily, the main type of relative ∞-category of interest in this article is right proper:

Proposition 1.3.6. *Let $\mathcal{E}$ be a locally contractible ∞-topos, then $\mathcal{E}$ together with its class W of shape equivalences is a right proper relative ∞-category.*

Proof. For every object E in $\mathcal{E}$ the functor $\mathcal{E}_{/E} \xleftarrow{E\times\_} \mathcal{E} \xleftarrow{\pi^*} \mathcal{S}$ is cocontinuous and left exact, and thus the constituent left adjoint of the unique geometric morphism $\mathcal{E}_{/E} \to \mathcal{S}$. This functor admits a left adjoint, exhibiting $\mathcal{E}_{/E}$ as locally contractible, with shape equivalences those maps in $\mathcal{E}_{/E}$ whose underlying map in $\mathcal{E}$ is a shape equivalence. Thus, by [Clo24b, Cor. 1.22] $(\pi_!)_{/E} : \mathcal{E}_{/E} \to \mathcal{S}_{/\pi_! E}$ is a localisation, exhibiting the canonical functor $W^{-1}(\mathcal{E}_{/E}) \to \mathcal{S}_{/\pi_! E}$ as an equivalence. □

We now introduce our main tool for recognising sharp morphisms in a relative ∞-category.

Definition 1.3.7. A ***fibration structure*** on (C, W) consists of a subcategory $\mathrm{Fib} \subseteq C$. The morphisms in Fib and $\mathrm{Fib} \cap W$ are called *fibrations* and *trivial fibrations* respectively. An object x for which some (and therefore any) morphism to a final object of C is a fibration is called *fibrant*. The triple (C, W, Fib) is required to satisfy the following conditions:

(a) Fib contains all equivalences in C.

(b) W satisfies the 2-out-of-3 property.

(c) In any diagram

$$\begin{tikzcd} & x' \arrow[d, "f"] \\ y \arrow[r] & x \end{tikzcd}$$

such that f is a fibration or trivial fibration, the pullback is again a fibration or trivial fibration, respectively.

(d) Any morphism $x \to y$ admits a factorisation $x \to x' \to y$ such that $x \to x'$ is a weak equivalence, and $x' \to y$ is a fibration.

An ∞-category equipped with a fibration structure is called a ***fibration ∞-category***.

Dually, a subcategory $\mathrm{Cof} \subseteq C$ is a ***cofibration structure*** on C if $\mathrm{Cof}^{\mathrm{op}}$ is a fibration structure on $(C^{\mathrm{op}}, W^{\mathrm{op}})$. An ∞-category equipped with a cofibration structure is called a ***cofibration ∞-category***. ⌟

Remark 1.3.8. Our notion of *fibration structure* is slightly stronger than the notion of *∞-category with weak equivalences and fibrations* considered in [Cis19, Def. 7.4.12]. ⌟

Example 1.3.9. The classes of weak equivalences and fibrations of any ∞-model category (see §B) form a fibration structure, which moreover satisfy the condition of Proposition 1.3.14 if it admits all limits. All fibration structures considered in this article will be of this form (however, see Remark 2.3.13). ⌟

From now on we assume that (C, W) is equipped with a fibration structure Fib.

Proposition 1.3.10. *Let*

$$\begin{tikzcd} y' \arrow[r] \arrow[d] & x' \arrow[d] \\ y \arrow[r, "f"] & x \end{tikzcd} \tag{7}$$

be pullback square in C, where

(a) *y and x are right proper,*

(b) *$x' \to x$ is a fibration, and*

(c) *for all morphisms $z \to x$ the pullback along $f : y \to x$ exists,*

then the square is a homotopy pullback.

The following proof is similar to the last part of Remark 1.3.5.

Proof. Equip the relative ∞-category $(C_{/y}, W_{/y})$ with the cofibration structure in which all morphisms are cofibrations, and $(C_{/x}, W_{/x})$ with the fibration structured induced by Fib.

First we observe that by [Cis19, Lm. 7.5.24] the functor $\mathbf{L}f_!$ is simply given by postcomposing with $f : y \to x$ in $W^{-1}C$, so we will again denote it by $f_!$.

We obtain a square

$$\begin{tikzcd} \mathbf{R}f^*x' \arrow[r] \arrow[d] \arrow[dr] & x' \arrow[d] \\ y \arrow[r] & x \end{tikzcd}$$

exhibiting $\mathbf{R}f^*x'$ as a pullback in $W^{-1}C$, where the lower triangle is given by $f_!$, and the upper triangle is given by the counit.

Denote by $p : \Lambda^2_2 \to C$ the restriction of (7) to Λ^2_2. We will construct an isomorphism $y' \to \mathbf{R}f^*x$ in $(W^{-1}C)_{/p}$. Observe that this corresponds to a functor $q : \Delta^3 \sqcup_{\Delta^{\{0,1,3\}}} \Delta^3 \to W^{-1}C$. We will first construct the restriction of q to the first copy of Δ^3, and then extend it to the second copy.

As $x' \to x$ is fibrant in $C_{/x}$ we obtain a canonical isomorphism $y' \xrightarrow{\simeq} \mathbf{R}f^*x'$ in $W^{-1}C_{/y}$. Applying $f_!$ produces the diagram $\Delta^2 \to W^{-1}C_{/x}$ given by

$$\begin{tikzcd} y' \arrow[r] \arrow[d] \arrow[dr] & \mathbf{R}f^*x' \arrow[d] \arrow[dl] \\ y \arrow[r] & x. \end{tikzcd}$$

Denote by $C^{\mathrm{fib}}_{/x}$ the full subcategory of $C_{/x}$ spanned by the fibrant objects. By the discussion in [Cis19, §7.5] the whiskering of $f_! \circ \mathbf{R}f^* \to \mathrm{id}$ with the composition of $C^{\mathrm{fib}}_{/x} \to W^{-1}(C^{\mathrm{fib}}_{/x}) \xrightarrow{\simeq} (W^{-1}C)_{/x}$ is canonically equivalent to the whiskering of $f_! \circ f^* \to \mathrm{id}$ with $C^{\mathrm{fib}}_{/x} \hookrightarrow C_{/x}$. Applying these equivalent natural transformations to

$$\begin{tikzcd} y' \arrow[d] \arrow[dr] & \\ \mathbf{R}f^*x' \arrow[r] & x \end{tikzcd}$$

produces the diagram

$$\begin{tikzcd}
y' \arrow[r] \arrow[d] \arrow[dr] & x' \arrow[d] \\
\mathbf{R}f^*x' \arrow[r] \arrow[ur] & x.
\end{tikzcd}$$

finishing the proof. □

Corollary 1.3.11. *Let (C, W, Fib) be a finitely complete fibration category such that (C, W) is right proper, then every fibration is sharp.* □

Corollary 1.3.12. *Let $\mathcal{E}$ be a locally contractible ∞-topos, and let Fib be a fibration structure on $\mathcal{E}$ w.r.t. the shape equivalences, then any fibration is sharp w.r.t. the shape equivalences.* □

The following simple proposition offers an effective method for detecting sharp morphisms.

Proposition 1.3.13. *Let $(C, W), (C', W')$ be relative ∞-categories with pullbacks, and let $f : C \to C'$ be a functor. Assume that*

(a) *f preserves pullbacks,*

(b) *$fW \subseteq W'$, and*

(c) *the induced functor $W^{-1}C \to W'^{-1}C'$ is an equivalence of ∞-categories,*

then any morphism $x \to y$ in C is sharp if $fx \to fy$ is.

Proof. Any pullback along $x \to y$ is sent to a pullback along $fx \to fy$ which is sent to a pullback in $W'^{-1}C'$, so that any pullback along $x \to y$ is sent to a pullback in $W^{-1}C$. □

The following result treats not-necessarily-finite homotopy limits.

Proposition 1.3.14 ([Cis19, Prop. 7.7.4])**.** *If an arbitrary product of fibrant objects in C is again fibrant, and an arbitrary product of trivial fibrations is again a trivial fibration, then arbitrary products of fibrant objects are homotopy products.* □

Remark 1.3.15. Model categories and ∞-categories are frequently viewed as competing foundations for homotopy theory (see [MO78400]). In reality, the axioms for model categories can be interpreted verbatim in the setting of ∞-categories (see §B), not just ordinary categories, and one observes that model structures are simply tools for studying localisations. Any ∞-category may be obtained as the localisation of an ordinary relative category (see [Cis19, Prop. 7.3.15], [BK12]), and any presentable ∞-category may be obtained as the localisation of a combinatorial simplicial model category (see [Lur09, Prop. A.3.7.6] & [Lur17, Th. 1.3.4.20] & [Cis19, Th. 7.5.18]). Before the work of Joyal, Simpson, Toën, Rezk, Lurie and many others, it was simply not practical to present ∞-categories in any other way than as ordinary relative categories (or a simplicially enriched categories). Nowadays, one has a *choice* of whether one wishes to work in a given ∞-category C, or whether one wishes to view C as the localisation of some other (∞-)category D. The optimal choice of D is not necessarily an ordinary category, as seen in Mazel-Gee's generalisation of the Goerss-Hopkins obstruction theorem (see [MG16]), and in our applications to differentiable sheaves in this article. ⌟

1.4 Constructing homotopical calculi in locally contractible (∞-)toposes

In §1.3 we saw how fibration structures are well suited to identifying homotopy limits in (subcategories of) locally contractible ∞-toposes; this subsection concerns their construction using test categories.

Throughout this subsection A denotes a small ordinary category. By [Clo24b, Ex. 1.23] the ∞-topos $[A^{\mathrm{op}}, \mathcal{S}]$ models the ∞-category $\mathcal{S}_{/A_{\simeq}}$ in the sense that taking colimits produces a localisation $[A^{\mathrm{op}}, \mathcal{S}] \to \mathcal{S}_{/A_{\simeq}}$. In the special case $A = \Delta$ something rather remarkable happens. The restriction of the functor $[\Delta^{\mathrm{op}}, \mathcal{S}] \to \mathcal{S}_{/\Delta_{\simeq}} \xrightarrow{\sim} \mathcal{S}$ to $\widehat{\Delta} \to \mathcal{S}$ is still a localisation, exhibiting the classical way in which homotopy types are modelled by simplicial sets. As the construction of the model category of simplicial sets is quite involved, one might expect this phenomenon to be particular to Δ, but it turns out to be surprisingly common. Conceptually, categories for which this phenomenon arise are precisely *test categories*.

The theory of test categories is outlined in §1.4.1, with a focus on how ordinary categories of set-valued presheaves on test categories model slice ∞-categories of $\mathcal{S}$. Then in §1.4.2 we discuss how to construct model structures on ∞-categories of homotopy-type-valued presheaves on test categories, which may then be transferred to locally contractible ∞-toposes via the nerves of [Clo24b, §1.2.1].

1.4.1 Test categories

The basic ideas discussed in this subsection are essentially all due to Grothendieck, and were first outlined in [Gro83]. A systematic account of Grothendieck's theory is given by Maltsiniotis in [Mal05]. The theory of test categories and test toposes, and in particular their model categorical aspects, are further developed in [Cis03] and [Cis06].

The starting point for understanding the phenomenon discussed in the introduction of §1.4 is the following fact: Recall that the classifying space of an ∞-category is nothing but the homotopy type obtained by inverting all its arrows, and furthermore, that the classifying space construction is left adjoint to the inclusion of $\mathcal{S}$ into **Cat**, the ∞-category of ∞-categories. Then, paralleling the situation for $[\Delta^{\mathrm{op}}, \mathcal{S}]$, the restriction of the classifying space functor $(\_)_{\simeq}$ to the $(2,1)$-category $\mathbf{Cat}_{(1,1)}$ of ordinary categories exhibits $\mathcal{S}$ as a localisation of $\mathbf{Cat}_{(1,1)}$:

$$\begin{tikzcd} \mathbf{Cat}_{(1,1)} \arrow[r, hook] & \mathbf{Cat} \arrow[r, shift left, "(\_)_{\simeq}"] \arrow[r, phantom, "\perp"] & \mathcal{S} \arrow[l, shift left, hook] \end{tikzcd}$$

The $(2,1)$-category $\mathbf{Cat}_{(1,1)}$ itself is the localisation of the ordinary category of ordinary categories $\mathbf{Cat}'_{(1,1)}$ (along the equivalences of categories).

The fact that $\mathcal{S}$ is a localisation of $\mathbf{Cat}'_{(1,1)}$ has been known in essence since [Ill72, Cor. 3.3.1] (specifically, that the category of elements of a simplicial set encodes the same homotopy type as the simplicial set itself is shown in [Ill72, Th. 3.3.ii]. Illusie attributes the ideas presented in [Ill72, §3.3] to Quillen; see also [Qui73]). Moreover, Thomasson shows that the relative category $\mathbf{Cat}'_{(1,1)}$ together with the weak equivalences induced by $(\_)_{\simeq}$ is right proper (see Definition 1.3.2), by exhibiting a right proper model structure on $\mathbf{Cat}'_{(1,1)}$ by right transferring the Kan-Quillen model structure (which is right proper) along the functor $\mathrm{Ex}^2 \circ N : \mathbf{Cat}'_{(1,1)} \to \widehat{\Delta}$ (see [Tho80]). Thus, the category $(\mathbf{Cat}'_{(1,1)})_{/A}$ is a model for $\mathcal{S}_{/A_{\simeq}}$; a model which turns out to be particularly convenient for determining conditions on A such that $\operatorname{colim} : \widehat{A} \to \mathcal{S}_{/A_\simeq}$ is a localisation. Then, $\operatorname{colim} : [A^{\mathrm{op}}, \mathcal{S}] \to \mathcal{S}$ factors as $[A^{\mathrm{op}}, \mathcal{S}] \xrightarrow{A_{/\_}} \mathbf{Cat} \xrightarrow{(\_)_\simeq} \mathcal{S}$, which restricts to $\widehat{A} \xrightarrow{A_{/\_}} \mathbf{Cat}'_{(1,1)} \xrightarrow{(\_)_\simeq} \mathcal{S}$. Thus, $A_{/\_}$ models the left adjoint of the adjunction $\operatorname{colim} : [A^{\mathrm{op}}, \mathcal{S}] \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathcal{S}$.

The functor $A_{/\_}$ also admits a right adjoint given by $N_A : C \mapsto (a \mapsto \operatorname{Hom}(A_{/a}, C))$. The category A is a ***weak test category*** if N_A sends all functors $C \to D$ such that $C_\simeq \to D_\simeq$ is an isomorphism to shape equivalences, and if the resulting adjunction $W^{-1}\widehat{A} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathcal{S}$ is an adjoint equivalence. We can now state the main definition of this subsection:

Definition 1.4.1. The category A is a ***local test category*** if $A_{/a}$ is a weak test category for all a in A. The category A is a ***test category*** if it is a local test category, and if $A_\simeq = 1$. ⌟

Theorem 1.4.2 ([Cis06, Cor. 4.4.20])**.** *If A is a local test category, then the composition of the functors $A_{/\_} : \widehat{A} \to (\mathbf{Cat}_{(1,1)})_{/A} \to \mathcal{S}_{A_\simeq}$ is a localisation of $\widehat{A}$ along the shape equivalences.* □

Definition 1.4.3. Let A be a small ordinary category, then a presheaf X on A is called ***locally aspherical*** if $(a \times X)_\simeq = 1$ for all $a \in A$. ⌟

One of the key features of local test categories is that they admit many characterisations, as seen in the following theorem.

Theorem 1.4.4 ([Mal05, Th. 1.5.6] & [Cis06, Thms. 1.4.3 & 4.1.19 & 4.2.15])**.** *The following are equivalent:*

(I) *A is a local test category.*

(II) *The subobject classifier of $\widehat{A}$ is locally aspherical.*

(III) *The category $\widehat{A}$ admits a locally aspherical separating interval (see [Clo24b, Def. 1.29]).*

(IV) *Any morphism in $\widehat{A}$ with the right lifting property against all monomorphisms is a shape equivalence.*

(V) *The category $\widehat{A}$ admits a (cofibrantly generated) model structure in which the weak equivalences are the shape equivalences, and the cofibrations are the monomorphisms.*

□

Proposition 1.4.5. *The following are equivalent:*

(I) *A is sifted (see* [Lur09, Def. 5.5.8.1]*).*

(II) *$A_{/\simeq} = 1$ and $(A_{/a \times a'})_\simeq = 1$ for all $a, a' \in A$.*

(III) *$\operatorname{colim} : \widehat{A} \to \mathcal{S}$ preserves finite products.*

Proof. The implication (I) $\implies$ (III) follows from [Lur09, Lm. 5.5.8.11], and (II) is a special case of (III), establishing (III) $\implies$ (II), and (II) $\implies$ (I) follows from applying [Lur09, Th. 4.1.3.1] to [Lur09, Def. 5.5.8.1]. □

Definition 1.4.6. The category A is a ***strict test category*** if it is a local test category and satisfies the equivalent conditions of Proposition 1.4.5. ⌟

Applying Theorem 1.4.4 to strict test categories yields the following recognition theorem.

Proposition 1.4.7. *Let A be a small category admitting finite products and a representable separating interval (see [Clo24b, Def. 1.29]) on $\widehat{A}$, then A is a strict test category.* □

In [Mal05, §1.8] Cisinski and Maltsiniotis develop more sophisticated tools for recognising strict test categories, and produces some surprising examples thereof, such as the monoid of increasing functions $\mathbf{N} \to \mathbf{N}$ (see [Mal05, Ex. 1.8.15]).

Test toposes We give a very brief introduction to the theory of local test toposes developed in [Cis03]. Throughout our discussion on test toposes, $\mathcal{E}$ denotes an ordinary topos generated under colimits by a set of contractible objects, by which we mean objects which have contractible shape in the *hypercompletion* of the ∞-topos associated to $\mathcal{E}$ (in the sense of [Lur09, Prop. 6.4.5.7]), which we denote by $\mathcal{E}_\infty$.

We begin with the following generalisation of Theorem 1.4.4, which we then use to give a definition of local test toposes.

Theorem 1.4.8 ([Cis03, Th. 4.2.8])**.** *The following are equivalent:*

(I) *For any object X in $\mathcal{E}$ the projection map $X \times \Omega_{\mathcal{E}} \to X$ is a shape equivalence.*

(II) *Any morphism in $\mathcal{E}$ with the right lifting property against all monomorphisms is a shape equivalence.*

(III) *There exists a subcategory of $\mathcal{E}$ spanned by objects of contractible shape, which is moreover a local test category and which generates $\mathcal{E}$ under colimits.*

(IV) *There exists a (necessarily unique as well as cofibrantly generated) model structure on $\mathcal{E}$ in which the weak equivalences are the shape equivalences, and in which the cofibrations are the monomorphisms.*

□

Definition 1.4.9. An ordinary topos satisfying the equivalent conditions of Theorem 1.4.8 is called a ***local test topos***. A local test topos with trivial shape is a ***test topos***. A test topos, whose shape functor commutes with finite products, is a ***strict test topos***. On any topos, the model structure given by Theorem 1.4.8 is referred to as the ***canonical model structure***. ⌟

Proposition 1.4.10. *Assume $\mathcal{E}$ is a local test topos, then $\mathcal{E}_\infty$ is locally contractible, and the composition $\mathcal{E} \hookrightarrow \mathcal{E}_\infty \xrightarrow{\pi_!} \mathcal{S}_{/\pi_!\mathbf{1}_{\mathcal{E}_\infty}}$ is a localisation.*

Proof. The proposition is equivalent to the statement that the inclusion $\mathcal{E} \hookrightarrow \mathcal{E}_\infty$ induces an equivalence of ∞-categories upon localising along shape equivalences. Let $C \subseteq \mathcal{E}$ be a subcategory satisfying (III) of Theorem 1.4.8. Consider the diagram

$$\begin{tikzcd}
{[C^{\mathrm{op}}, \mathcal{S}]} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{E}_\infty \arrow[l, shift left] \\
\widehat{C} \arrow[u, hook] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[u, hook] \arrow[l, shift left]
\end{tikzcd}$$

then the top adjunction is a geometric embedding by [Lur18, Cor. 20.4.3.3 & Prop. 20.4.5.1], and a local shape equivalence by [Clo24b, Prop. 1.11], so that the right adjoint is shape preserving by [Clo24b, Prop. 1.12]. Thus the unit and counit are natural weak equivalences, and the same is true of the bottom adjunction, as it is a restriction of the top one, so that by [Cis19, Prop. 7.1.14] both adjunctions descend to equivalences of ∞-categories upon localising. The left vertical functor induces an equivalence upon localising by Theorem 1.4.2, so that the right vertical functor induces an equivalence upon localising by the 2-out-of-3 property. ☐

Remark 1.4.11. Proposition 1.4.10 fails if we do not assume that $\mathcal{E}_\infty$ is hypercomplete, because then $\mathcal{E}_\infty$ may no longer be generated by objects in C under colimits (see [Ane]). ⌟

Lemma 1.4.12. *Let X be an object of $\mathcal{E}_\infty$, then $(\mathcal{E}_\infty)_{/X}$ is equivalent to the hypercompletion of the ∞-topos associated to $\mathcal{E}_{/X}$.*

Proof. Denote by $\mathcal{F}$ the ∞-topos universally associated to $\mathcal{E}$ (see [Lur09, Prop. 6.4.5.7]), and by $(\mathcal{E}_{/X})_\infty$ the hypercompletion of the ∞-topos universally associated to $\mathcal{E}_{/X}$ then we obtain a commutative square

$$\begin{tikzcd}
\mathcal{F}_{/X} \arrow[d] & (\mathcal{E}_{/X})_\infty \arrow[l, hook'] \arrow[d] \\
\mathcal{F} & \mathcal{E}_\infty \arrow[l, hook']
\end{tikzcd}$$

By the universal property of $\mathcal{F}_{/X}$ (see [Lur09, Rmk. 6.3.5.8]) we may exhibit $(\mathcal{E}_{/X})_\infty$ as a subcategory of $(\mathcal{E}_\infty)_{/X}$. Conversely, $(\mathcal{E}_\infty)_{/X}$ is hypercomplete by [Lur09, Th. 6.5.3.12], so by the universal property of $(\mathcal{E}_{/X})_\infty$ the ∞-topos $(\mathcal{E}_\infty)_{/X}$ is a subcategory of $(\mathcal{E}_{/X})_\infty$. ☐

Proposition 1.4.13 ([Cis03, Cor. 5.3.20 & Cor. 4.2.12])**.** *Any local test topos — viewed as a relative category with its shape equivalences as weak equivalences — is proper.*

Proof. By Lemma 1.4.12 the composition of the functors $\mathcal{E}_{/X} \hookrightarrow (\mathcal{E}_\infty)_{/X} \xrightarrow{\pi_!} \mathcal{E}_{/\pi_! X}$ is a localisation. ☐

We finish with an application of Theorem 1.4.8 to equivariant homotopy theory.

Theorem 1.4.14. *Assume that $\mathcal{E}$ is a strict test topos, and that G is a group object in $\mathcal{E}$, then $\mathcal{E}_G$ is a test topos. A morphism in $\mathcal{E}_G$ is a shape equivalence iff its underlying morphism in $\mathcal{E}$ is, and the induced functor $\mathcal{E}_G \to \mathcal{S}_{\pi_! G}$ is a localisation along the shape equivalences in $\mathcal{E}_G$.*

Proof. From the equivalence of ∞-categories $(\mathcal{E}_\infty)_G = (\mathcal{E}_\infty)_{/BG}$ we see that $\mathcal{E}_G$ is equivalent to $\big((\mathcal{E}_\infty)_{/BG}\big)_{\leq 0}$. Let C be a small subcategory of $\mathcal{E}$ spanned by objects of contractible shape generating $\mathcal{E}$ (and thus $\mathcal{E}_\infty$) under colimits, then $C_{/BG}$ is an ordinary category whose objects are of contractible shape and generate $(\mathcal{E}_\infty)_{/BG}$ under colimits. We will check that $\big((\mathcal{E}_\infty)_{/BG}\big)_{\leq 0}$ satisfies (II) of Theorem 1.4.8, verifying the first part of the theorem. Let $X \to Y$ be a morphism in $\big((\mathcal{E}_\infty)_{/BG}\big)_{\leq 0}$ lifting against all monomorphisms, then the underlying morphism of $X \to Y$ in $\mathcal{E}$ lifts against all monomorphisms (and is thus a shape equivalence), as any lifting problem against $X \to Y$ in $\mathcal{E}$ may be promoted to one in $\mathcal{E}_{/BG}$ by composing with the morphism $Y \to BG$.

Next, the induced functor $\mathcal{E}_G \to \mathcal{S}_{\pi_! G}$ is a localisation by the following diagram and the fact that

$(\mathcal{E}_\infty)_{/BG} = ((\mathcal{E}_\infty)_{/BG})_{\le 0})_\infty$ by Lemma 1.4.12:

$$\begin{tikzcd}
\mathcal{E}_G \arrow[r, hook] \arrow[d, "\simeq"'] & (\mathcal{E}_\infty)_G \arrow[r] \arrow[d, "\simeq"'] & \mathcal{S}_{\pi_! G} \arrow[d, "\simeq"'] \\
\big((\mathcal{E}_\infty)_{/BG}\big)_{\le 0} \arrow[r, hook] & (\mathcal{E}_\infty)_{/BG} \arrow[r] & \mathcal{S}_{/B\pi_! G}
\end{tikzcd}$$

Finally, a morphism $X \to Y$ in $\mathcal{E}_G$ is a shape equivalence iff $(\pi_\mathcal{E})_! X \to (\pi_\mathcal{E}) Y$ is an isomorphism in $\mathcal{S}_{\pi_! G}$, iff $(\pi_\mathcal{E})_! X \to (\pi_\mathcal{E}) Y$ is an isomorphism in $\mathcal{S}$, iff the underlying morphism of $X \to Y$ in $\mathcal{E}$ is a shape equivalence. □

1.4.2 Transferring model structures to locally contractible (∞-)toposes

Here we finally construct model structures on locally contractible ∞-toposes and test toposes for which the weak equivalences are the shape equivalences. We begin by recalling some basic theory of cofibrantly generated model ∞-categories, in particular, two theorems on constructing and transferring cofibrantly generated model structures, respectively, which are classical in the ordinary categorical setting. Then, for any local test category A we extend the canonical model structure on $\widehat{A}$ to $[A^{\mathrm{op}}, \mathcal{S}]$. Finally, we transfer the model structure on $[A^{\mathrm{op}}, \mathcal{S}]$ to locally contractible ∞-toposes, and the model structure on $\widehat{A}$ to test toposes.

Definition 1.4.15. A complete and cocomplete model ∞-category M is ***cofibrantly generated*** if there exist *sets* I, J of morphisms in M such that

1. $C = {}^\boxslash(I^\boxslash)$,
2. $C \cap W = {}^\boxslash(J^\boxslash)$, and
3. I and J permit the small object argument (see [MG14, §3.5]).

⌟

Definition 1.4.16. Let M be a cofibrantly generated model ∞-category, then a ***relative I-complex*** (resp. ***J-complex***) is any morphism which can be written as the transfinite composition (see [DAG X, Def. 1.4.2]) of pushouts of morphisms in I (resp. J). ⌟

By [DAG X, Prop. 1.4.7] any *set* of morphisms in a presentable ∞-category admits the small object argument.

Warning 1.4.17. Let I be a set of morphisms in an ∞-category C satisfying the small object argument, then the attendant factorisation of any morphism in C into a relative I-complex followed by a morphism in $I^\boxslash$ is *not* functorial. See [DAG X, Warning 1.4.8] and [MG14, Rmk. 3.7]. ⌟

Proposition 1.4.18. *Let M be a presentable ∞-category, let $W \subseteq M$ be a subcategory, which is closed under retracts, and satisfies the 2-out-of-3 property. Suppose that I and J are* sets *of homotopy classes of maps such that*

(a) ${}^\boxslash(J^\boxslash) \subseteq {}^\boxslash(I^\boxslash) \cap W$

(b) $I^{\boxslash} \subseteq J^{\boxslash} \cap W$

(c) *and either*

(c$_1$) ${}^{\boxslash}(J^{\boxslash}) = {}^{\boxslash}(I^{\boxslash}) \cap W$, *or*

(c$_2$) $I^{\boxslash} = J^{\boxslash} \cap W$,

then the I and J define a cofibrantly generated model structure on M whose weak equivalences are W.

Proof. In either case, by [DAG X, Prop. 1.4.7] the pairs $({}^{\boxslash}(J^{\boxslash}), {}^{\boxslash}(I^{\boxslash}) \cap W)$ and $(I^{\boxslash}, J^{\boxslash} \cap W)$ satisfy the conditions of Proposition B.0.10. □

Proposition 1.4.19. *Let M be a cofibrantly generated model ∞-category with generating cofibrations I and generating trivial cofibrations J, let N be a presentable ∞-category, and consider an adjunction $f : M \rightleftarrows N : u$. If the functor u takes relative fJ-cell complexes to weak equivalences, then*

(1) *the ∞-category N admits a cofibrantly generated model structure whose weak equivalences are those morphisms carried to weak equivalences by u, and with generating cofibrations and trivial cofibrations given by fI and fJ respectively, and*

(2) *the adjunction $f : M \rightleftarrows N : u$ is a Quillen adjunction.*

Proof. The condition in the proposition precisely ensures that fI and fJ satisfy (a) of Proposition 1.4.18, and the two conditions (b) and (c$_1$) are satisfied by Proposition B.0.6. □

We can now extend the canonical model structure. The following proposition generalises [MG14, Th. 4.4].

Proposition 1.4.20. *Let A be a local test category, then there exists a (necessarily unique) cofibrantly generated model structure on $[A^{\mathrm{op}}, \mathcal{S}]$ whose weak equivalences are the shape equivalences, and whose trivial fibrations are characterised by having the right lifting property against the monomorphisms in $\widehat{A}$.*

Furthermore, if I and J are generating cofibrations and trivial cofibrations, respectively, of the canonical model structure on $\widehat{A}$, then these generate the model structure on $[A^{\mathrm{op}}, \mathcal{S}]$.

Proof. Let I and J be generating cofibrations and trivial cofibrations, respectively, of the canonical model structure on $\widehat{A}$. Any morphism $X \to Y$ which lifts against all monomorphisms in $\widehat{A}$ clearly lifts against I. Conversely, assume that $X \to Y$ lifts against I. Any monomorphism may be constructed as a retract of an I-cellular map which by Lemmas 1.1.1 - 1.1.4 is again a morphism in $\widehat{A}$, so that $X \to Y$ lifts against all monomorphisms between objects in $\widehat{A}$ by [DAG X, Cor. 1.4.10].

We will now verify that the set of shape equivalences W together with I, J satisfy (a), and (b), (c$_2$) of Proposition 1.4.18.

Proof of (a): By Lemmas 1.1.1 - 1.1.4 all colimits involved in constructing the morphisms in ${}^{\boxslash}(J^{\boxslash})$ are homotopy colimits. As all morphisms in J are weak equivalences, the morphisms in ${}^{\boxslash}(J^{\boxslash})$ must be weak equivalences.

Proof of (b): The inclusion $I^{\boxslash} \subseteq J^{\boxslash}$ is clear as $J \subseteq {}^{\boxslash}(I^{\boxslash})$, so we need to show $I^{\boxslash} \subseteq W$. So, let $X \to Y$ be a morphism in $I^{\boxslash}$.

First, we show that it is enough to prove the statement in the case when Y is representable. For all objects a in A, and all maps $a \to Y$ the morphism $a \times_Y X \to a$ is in $I^{\boxslash}$. If these morphisms are in W, then $X \to Y$ is in W by faithful descent, as the morphism can be written as a colimit indexed by $A_{/Y} \to A$.

So, assume that Y is representable. As a morphism in $A_{/Y}$ is a monomorphism iff it is a monomorphism in A, we may furthermore assume that A has a final object, and that Y is such a final object.

As the shape of the presheaf represented by the final object in A is contractible, it is enough to show that the shape of X is contractible. Now, the shape of X is given by $(A_{/X})_{\simeq} \simeq \operatorname{Ex}^\infty A_{/X}$, so that any map $S^k \to \pi_! X$ $(k \geq 0)$ may be represented by a map $\operatorname{Sd}^n \partial\Delta^k \to A_{/X}$ for some $n \geq 0$. If $n \geq 1$, then Sd^n is a finite poset, and therefore a finite direct category. We will show that for any finite direct category I and any functor $I \to A_{/X}$ we obtain a factorisation

$$\begin{tikzcd} I_{\simeq} \arrow[r] \arrow[d] & (A_{/X})_{\simeq} \\ * \arrow[ur, dashed] & \end{tikzcd} \tag{8}$$

Consider the diagram $f : I \to A$, and take a Reedy cofibrant replacement $\tilde{f} \xrightarrow{\sim} f$ in $\widehat{A}$ (see [Cis19, Prop. 7.4.19]), then by an inductive application of [Cis19, Cor. 7.4.4] and Lemmas 1.1.1 & 1.1.2 we see that the colimit of $\tilde{f}$ is 0-truncated. The map $I_{\simeq} \to (A_{/X})_{\simeq}$ corresponds to the map $\pi_! \operatorname{colim} \tilde{f} \to \pi_! X$. Consider a factorisation $\operatorname{colim} \tilde{f} \to c \to 1$ in $\widehat{A}$, where $\operatorname{colim} \tilde{f} \to c$ is a monomorphism, and $c \to 1$ is a trivial fibration, and thus a weak equivalence. By our assumption on X, we obtain a lift

$$\begin{tikzcd} \operatorname{colim} \tilde{f} \arrow[r] \arrow[d, hook] & X \\ c \arrow[ur, dashed] & \end{tikzcd}$$

Taking the shape of this diagram yields the desired lift in (8).

Proof of (c_2): The proof of this fact for $A = \Delta$ is given in [MG14, Prop. 7.9], and may be interpreted verbatim in our setting. □

Combining Propositions 1.3.6 & 1.3.10 yields:

Proposition 1.4.21. *Let A be a local test category, then any fibration in the model structure on $[A^{\mathrm{op}}, \mathcal{S}]$ constructed in Proposition 1.4.20 is sharp.* □

We now construct model structures on locally contractible ∞-toposes and on test toposes. Both of these theorems should be compared to [Clo24b, Th. 1.27].

Proposition 1.4.22. *Let*

(i) *$\mathcal{E}$ be an ∞-topos, generated under colimits by a small subcategory C consisting of contractible objects (so that $\mathcal{E}$ is locally contractible),*

(ii) *A, a small ∞-category, and*

(iii) *$u : A \to C$, a functor.*

Assume that

(a) *$u : A \to C$ is initial, and that*

(b) *$[A^{\mathrm{op}}, \mathcal{S}]$ admits a cofibrantly generated model structure with sets I and J of, respectively, generating cofibrations and generating trivial cofibrations, and in which the weak equivalences are the shape equivalences,*

there exists a cofibrantly generated model structure on $\mathcal{E}$ such that

(1) *the weak equivalences are precisely the shape equivalences,*

(2) *the sets $u_!I$ and $u_!J$ are generating sets for the cofibrations and trivial cofibrations, respectively, and*

(3) *the adjunction $u_! : [A^{\mathrm{op}}, \mathcal{S}] \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathcal{E} : u^*$ is a Quillen equivalence.*

If moreover

(c) *the inclusions $u\ell \hookrightarrow ud$ admit retracts for all morphisms $\ell \hookrightarrow d$ in J,*

then

(4) *all objects in the resulting model structure on $\mathcal{E}$ are fibrant.*

Proof. We will use Proposition 1.4.19 to transfer the model structure on $[A^{\mathrm{op}}, \mathcal{S}]$ to $\mathcal{E}$. By [Clo24b, Th. 1.27] the weak equivalences in $\mathcal{E}$ created by u^* are precisely the shape equivalences. The condition in the statement of Proposition 1.4.19 is then trivially satisfied, because the shape functor $\pi_! : \mathcal{E} \to \mathcal{S}$ commutes with all colimits, so that we obtain a Quillen adjunction, which is a Quillen equivalence, again by [Clo24b, Th. 1.27]. Conclusion (4) is obvious. □

Theorem 1.4.23. *Let*

(i) *$\mathcal{E}$ be an ∞-topos, generated under small colimits by a small subcategory C of $\mathcal{E}_{\leq 0}$ consisting of contractible objects,*

(ii) *A, a local test category, and*

(iii) *$u : A \to C$, a functor.*

Assume that

(a) *$u : A \to C$ is initial,*

(b) *$u_! : [A^{\mathrm{op}}, \mathcal{S}] \to \mathcal{E}$ preserves 0-truncated objects, and*

(c) *$u_! : \widehat{A} \to \mathcal{E}_{\leq 0}$ preserves monomorphisms,*

then for any sets I and J of, respectively, generating cofibrations and generating trivial cofibrations for the canonical model structure on $\widehat{A}$, there exists a cofibrantly generated model structure on $\mathcal{E}_{\leq 0}$ such that

(1) *the weak equivalences are precisely the shape equivalences,*

(2) *the sets $u_!I$ and $u_!J$ are generating sets for the cofibrations and trivial cofibrations, respectively, and*

(3) *the adjunction* $u_! : \widehat{A} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{E}_{\leq 0} : u^*$ *is a Quillen equivalence.*

If moreover

(e) *the inclusions $u\ell \hookrightarrow ud$ admit retracts for all morphisms $\ell \hookrightarrow d$ in J,*

then

(4) *all objects in the resulting model structure on $\mathcal{E}_{\leq 0}$ are fibrant.*

The proof of Theorem 1.4.23 is very similar to the proof of Proposition 1.4.22.

Proof. The shape equivalences in $\mathcal{E}_{\leq 0}$ are created by u^* by [Clo24b, Th. 1.27]. The conditions of Proposition 1.4.19 are satisfied by assumptions (b) & (c) and Corollary 1.1.5, so that $u_! \dashv u^*$ is a Quillen adjunction. By (b) the unit and counit of the $u_! : \widehat{A} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{E}_{\leq 0} : u^*$ coincide with the ones of $u_! : [A^{\mathrm{op}}, \mathcal{S}] \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{E} : u^*$, so that $u_! \dashv u^*$ is a Quillen equivalence by Proposition 1.4.22.

Conclusion (4) is obvious. □

We conclude this section with a discussion of some criteria for checking conditions (a) - (c) in Theorem 1.4.23. We have already seen that condition (a) may be checked using [Clo24b, Props. 1.32 & 1.33]. We add two simple criteria for verifying (b) & (c) of Theorem 1.4.23 in the case when $A = \Delta, \square$.

Proposition 1.4.24. *Let $\mathcal{E}$ be an ∞-topos, and $u : \Delta \to \mathcal{E}_{\leq 0}$ a functor, and assume that the unique cocontinuous extension $u_! : [\Delta^{\mathrm{op}}, \mathcal{S}] \to \mathcal{E}$ carries*

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \Delta^{\{1\}} \arrow[d] \\ \Delta^{\{0\}} \arrow[r] & \Delta^1 \end{tikzcd}$$

to a pullback, then

(1) $u_! : [\Delta^{\mathrm{op}}, \mathcal{S}] \to \mathcal{E}$ preserves 0-truncated objects, and the restricted functor

(2) $u_! : \widehat{\Delta} \to \mathcal{E}_{\leq 0}$ preserves monomorphisms.

Proof. By [Cis06, Lm. 2.1.9] and the assumption in the statement of the proposition, the Čech nerve of the map $\coprod_{i=0}^n u_!\Delta^{n-1} \xrightarrow{(d_0,\ldots,d_n)} u_!\Delta^n$ is given by the image under $u_!$ of the Čech nerve of $\coprod_{i=0}^n \Delta^{n-1} \xrightarrow{(d_0,\ldots,d_n)} \Delta^n$, so that $u_!\partial\Delta^n \to u_!\Delta^n$ is monomorphism for all $n \geq 0$. Then (1) follows from Propositions 1.1.1 - 1.1.3 and the way in which $u_!X$ is constructed via cell attachments for any simplicial set X. Finally, (2) follows from the fact that the monomorphism $X \to Y$ in $\widehat{\Delta}$ is obtained via a sequence of cell attachments, and the fact that monomorphisms are preserved under pushouts and filtered colimits. □

The proof of the following proposition is the same as the previous proof, except that it relies on [Cis06, Lm. 8.4.18] instead of [Cis06, Lm. 2.1.9].

Proposition 1.4.25. *Let $\mathcal{E}$ be an ∞-topos, and $u : \square \to \mathcal{E}_{\leq 0}$ a functor, and assume that the unique cocontinuous extension $u_! : [\square^{\mathrm{op}}, \mathcal{S}] \to \mathcal{E}$ carries $(\delta_i^0, \delta_i^1) : \square^{n-1} \sqcup \square^{n-1} \hookrightarrow \square^n$ to a monomorphism for all $n \geq i \geq 1$, then*

(1) $u_! : [\square^{\mathrm{op}}, \mathcal{S}] \to \mathcal{E}$ preserves 0-truncated objects, and the restricted functor

(2) $u_! : \widehat{\square} \to \mathcal{E}_{\leq 0}$ preserves monomorphisms.

□

The asymmetry between Propositions 1.4.24 & 1.4.25 disappears in the following situation:

Corollary 1.4.26. *Let $\mathcal{E}$ be an ∞-topos, and $u : \square \to \mathcal{E}_{\leq 0}$, a monoidal functor, and assume that the unique cocontinuous extension $u_! : [\square^{\mathrm{op}}, \mathcal{S}] \to \mathcal{E}$ carries*

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \square^{\{1\}} \arrow[d] \\ \square^{\{0\}} \arrow[r] & \square^1 \end{tikzcd}$$

to a pullback, then

(1) $u_! : [\square^{\mathrm{op}}, \mathcal{S}] \to \mathcal{E}$ preserves 0-truncated objects, and the restricted functor

(2) $u_! : \widehat{\square} \to \mathcal{E}_{\leq 0}$ preserves monomorphisms.

□

Proof. By assumption the morphism $(\delta_1^0, \delta_1^1) : \square^0 \sqcup \square^0 \hookrightarrow \square^1$ is carried to a monomorphism, and the maps $(\delta_i^0, \delta_i^1) : \square^{n-1} \sqcup \square^{n-1} \hookrightarrow \square^n$ may be rewritten as $\mathrm{id}_{\square^{i-1}} \times (\delta_1^0, \delta_1^1) \times \mathrm{id}_{\square^{n-i}}$, so the corollary follows from Proposition 1.4.25. □

2 Homotopical calculi on differentiable sheaves

Fix an element r of $\mathbf{N} \cup \{\infty\}$ for the remainder of this article. Recall that $\mathbf{Mfd}^r$ denotes the category of C^r-manifolds and C^r-maps, $\mathbf{Cart}^r$ the full subcategory spanned by $\mathbf{R}^d$ ($d \geq 0$), and that $\mathbf{Diff}^r$ denotes, equivalently, the ∞-topos of sheave on $\mathbf{Mfd}^r$ or $\mathbf{Cart}^r$ w.r.t. the open covering topology (see [Clo24a, Prop. 2.3]).

In §2.2 we exploit the technology of §1.4.2 to construct several model structures on $\mathbf{Diff}^r$ and related (∞-)categories, and discuss some of their properties. Then, in §2.3 we single out one of these model structures, the *Kihara model structure*, and use it to prove Theorem 2.3.28 which states that a large class of (possibly infinite dimensional) manifolds satisfies the *smooth Oka principle*. We will spend the rest of this introduction explaining what the differentiable Oka principle is, why it is interesting, and our strategy for proving Theorem 2.3.28.

In the present discussion all topological spaces are assumed to belong to some convenient category such as compactly or Δ-generated spaces, and $\mathbf{TSpc}$ denotes the category of such spaces. Let A, X be topological spaces with A, a CW complex, then $\mathbf{TSpc}(A, X)$ together with the compact open topology (denoted by $\underline{\mathbf{TSpc}}(A, X)$) is a model for $\mathcal{S}(LA, LX)$, where $L : \mathbf{TSpc} \to \mathcal{S}$ is the localisation functor.

This follows from the fact that the model structure on **TSpc** is Cartesian, by which $A \times \_ \dashv \underline{\mathbf{TSpc}}(A, \_)$ is a Quillen adjunction. As all objects in **TSpc** are fibrant, both $A \times \_$ and $\underline{\mathbf{TSpc}}(A, \_)$ preserve weak equivalences, and $A \times \_ \dashv \underline{\mathbf{TSpc}}(A, \_)$ descends to an adjunction on homotopy categories by [Cis19, Prop. 7.1.14]:

$$\begin{tikzcd} \mathbf{TSpc} \arrow[r, shift left, "A\times\_"] \arrow[d] \arrow[r, phantom, "\perp"] & \mathbf{TSpc} \arrow[l, shift left, "\underline{\mathbf{TSpc}}(A,\_)"] \arrow[d] \\ \mathcal{S} \arrow[r, shift left, "LA\times\_"] \arrow[r, phantom, "\perp"] & \mathcal{S}. \arrow[l, shift left, "\mathcal{S}(LA,\_)"] \end{tikzcd}$$

The derived left adjoint is given by $LA \times \_$ by Corollary 1.3.11, and thus the derived right adjoint must be canonically equivalent to $\mathcal{S}(LA, \_)$.

Moving on to the differentiable setting, let M be a closed smooth manifold, and N an arbitrary smooth manifold, then the set of smooth maps $\mathbf{Diff}^\infty(M, N)$ admits a canonical structure of a Fréchet manifold (see [GG73, Th. 1.11]). Via smoothing theory it is then possible to show that the homotopy type of this Fréchet manifold is equivalent to the homotopy type of $\underline{\mathbf{TSpc}}(M, N)$ (where M, N now denote the underlying topological spaces of the smooth manifolds M, N), which is equivalent to $\mathcal{S}(LM, LN)$, which is equivalent to $\mathcal{S}\big((\pi_{\mathbf{Diff}^\infty})_! M, (\pi_{\mathbf{Diff}^\infty})_! N\big)$ by [Clo24b, Th. 2.18]. By [Wal12, Lm A.1.7] the Fréchet manifold of smooth maps from M to N is canonically equivalent to $\underline{\mathbf{Diff}}^\infty(M, N)$, so it is natural to ask for which differentiable sheaves the internal mapping sheaf $\pi_! \underline{\mathbf{Diff}}^\infty(A, X)$ is a model for $\mathcal{S}(\pi_! A, \pi_! X)$. More precisely (and from now on for r no longer necessarily equal to ∞), by [Clo24b, Cor. 2.6] the shape functor $\pi_! : \mathbf{Diff}^r \to \mathcal{S}$ commutes with finite products so that we obtain a canonical map $\pi_! \underline{\mathbf{Diff}}^r(A, X) \to \mathcal{S}(\pi_! A, \pi_! X)$ by applying $\pi_!$ to the evaluation map $\underline{\mathbf{Diff}}^r(A, X) \times A \to X$, and then taking the transpose of $\pi_! \underline{\mathbf{Diff}}^r(A, X) \times \pi_! A \to \pi_! X$.

Definition 2.0.1. A differentiable sheaf A satisfies the ***differentiable Oka principle*** or is ***Oka cofibrant*** if for every r-times differentiable sheaf X the map $\pi_! \underline{\mathbf{Diff}}^r(A, X) \to \mathcal{S}(\pi_! A, \pi_! X)$ is an equivalence. ⌟

Remark 2.0.2. This terminology is inspired by work of Sati and Schreiber (e.g., [SS21]), where an object in $\mathbf{Diff}^\infty$ satisfying the differentiable Oka principle is said to satisfy the *smooth* Oka principle. We have chosen the term *differentiable* over *smooth* to emphasise that in our setting r is not necessarily equal to ∞. ⌟

In Theorem 2.3.28 we prove that a large class of (possibly infinite dimensional) manifolds satisfies the differentiable Oka principle for $r = \infty$ (see Remark 2.3.24). We will now discuss our proof strategy: Having constructed several model structures on $\mathbf{Diff}^r$ in §2.2, we might hope to prove Theorem 2.3.3 by showing that one of these satisfies the following three properties, so that we may argue similarly as in **TSpc**:

1. The model structure is Cartesian closed.

2. All objects are fibrant.

3. All manifolds are cofibrant.

Unfortunately, we are not able to get 1. and 2. simultaneously for any "reasonable" model structure, by Proposition 2.2.11.

We thus bring the theory of §1.3 to bear on our problem, which will allow us to think about homotopical calculi in a more flexible manner than allowed by model structures. Let us assume that we have already shown that a given differentiable sheaf A is Oka cofibrant, and that $S \to D$ is a map between Oka cofibrant objects, which we think of as constituting a "cell inclusion". Then, if we attach our "cell" D along a map $f : S \to A$, a natural way of showing that $A \cup_f D$ is also cofibrant is to show that the pullback

$$\begin{tikzcd} \underline{\mathbf{Diff}}^r(A \cup_f D, X) \arrow[r] \arrow[d] & \underline{\mathbf{Diff}}^r(D, X) \arrow[d] \\ \underline{\mathbf{Diff}}^r(A, X) \arrow[r] & \underline{\mathbf{Diff}}^r(S, X) \end{tikzcd}$$

is a homotopy pullback. Thus, we would like to find morphisms $S \to D$ between objects satisfying the differentiable Oka principle such that the morphism $\underline{\mathbf{Diff}}^r(D, X) \to \underline{\mathbf{Diff}}^r(S, X)$ is sharp for every differentiable sheaf X.

Definition 2.0.3. A morphism $S \to D$ is called an ***Oka cofibration*** if $\underline{\mathbf{Diff}}^r(D, X) \to \underline{\mathbf{Diff}}^r(S, X)$ is sharp for every differentiable sheaf X. ⌟

Kihara's simplices $\Delta^\bullet : \Delta \to \mathbf{Diff}^r$ (see [Kih19, § 1.2] or [Clo24b, Def. 2.12]) induce one of the model structures discussed in §2.2, and our strategy is then to show that, while the Kihara model structure is not Cartesian closed, the Kihara horn inclusions are Oka cofibrations. To accomplish this we introduce a new class of sharp morphisms, the *squishy fibrations* in §2.3.1, and show that the morphism $\underline{\mathbf{Diff}}(\Delta^n, X) \to \underline{\mathbf{Diff}}(\Lambda^n_k, X)$ is a squishy fibration for all horn inclusions and all r-times differentiable sheaves X. Then in §2.3.3 we show that a large class of (possibly infinite dimensional) *smooth* manifolds are Oka cofibrant by relating them to simplicial complexes built using Kihara's simplices, thus proving Theorem 2.3.28. We should like to emphasise that we are offering a general method of proving Theorem 2.3.28 and it would be interesting to discover other classes of Oka cofibrations between Oka cofibrant objects exhibiting manifolds (or other interesting differentiable sheaves) as Oka cofibrant.

Before turning to the proof of Theorem 2.3.28 we recall some basic facts about diffeological spaces in §2.1. Then in §2.2 we construct several model structures on $\mathbf{Diff}^\infty$ and various subcategories, whose weak equivalences are the shape equivalences, and which all model $\mathcal{S}$. Moreover, we give a new conceptual proof that the Quillen adjunction $|\_| : \widehat{\Delta} \rightleftarrows \mathbf{TSpc} : s$ is a Quillen equivalence.

2.1 Recollections on diffeological spaces

Diffeological spaces are particularly nice (0-truncated) differentiable sheaves, which have a good notion of underlying set. We bring the theory developed in §1.2 to bear on diffeological spaces in order to elucidate their basic properties and we briefly discuss the classification of diffeological principal bundles in §2.1.1.

Observe that since $\mathbf{Diff}^r$ is local, so is $\mathbf{Diff}^r_{\leq 0}$.

Definition 2.1.1. A ***diffeological space*** X is a concrete object in $\mathbf{Diff}^r_{\leq 0}$. A ***plot*** of X is a map $\mathbf{R}^n \to X$. The collection of all plots of X is called the ***diffeology*** of X. ⌟

Convention 2.1.2. Let X be a diffeological space, then plots of X are usually identified with their images under π_*. ⌟

A diffeological space is thus a set S together with a specified set of maps $\pi_*\mathbf{R}^d \to S$ for each $d \geq 0$, which are closed under precomposition of C^r-maps $\mathbf{R}^{d'} \to \mathbf{R}^d$, and such that the resulting presheaf on $\mathbf{Cart}^r$ is a sheaf.

Remark 2.1.3. A monomorphism $X \hookrightarrow Y$ of diffeological spaces is an embedding (see Definition 1.2.6) whenever for all $d \geq 0$, any map $\pi_*\mathbf{R}^d \to \pi_*X$ is a plot iff its composition with $\pi_*X \hookrightarrow \pi_*Y$ is. ⌟

Definition 2.1.4. Let Y be a diffeological space, and $X \subseteq \pi_*Y$, a subset, then the ***subspace diffeology*** on X is the unique diffeology on X in which a map $\pi_*\mathbf{R}^n \to \pi_*X$ is a plot iff it is a plot viewed as a map to Y. ⌟

Thus, the subspace diffeology on X is the unique diffeology making the inclusion $X \subseteq Y$ into an embedding.

Example 2.1.5. The standard simplex Δ^n with the subspace diffeology inherited from $\mathbf{R}^{n+1}$ is denoted by Δ^n_{sub}, and is referred to as the ***closed*** n***-simplex***. ⌟

Proposition 2.1.6 ([Wat12, Lm. 2.64])**.** *Write* $\mathbf{R}^n_+ := \left\{ (x_1, \ldots, x_n) \in \mathbf{R}^n \;\middle|\; x_1, \ldots, x_n \geq 0 \right\}$*, and endow this set with the subspace diffeology inherited from* $\mathbf{R}^n$*. A map* $f : \mathbf{R}^n_+ \to \mathbf{R}$ *is smooth iff it is the restriction of a smooth map* $U \to \mathbf{R}$*, where* U *is an open neighbourhood of* $\mathbf{R}^n_+$ *in* $\mathbf{R}^n$*.*

Proof. Write $s : \mathbf{R}^n \to \mathbf{R}^n$, $(x_1, \ldots, x_n) \mapsto (x_1^2, \ldots, x_n^2)$, then $f \circ s : \mathbf{R}^n \to \mathbf{R}$ is smooth, and moreover invariant under the action $(\mathbf{Z}^\times)^n \times \mathbf{R}^n \to \mathbf{R}^n$, $\big((\sigma_1, \ldots, \sigma_n), (x_1, \ldots, x_n)\big) \mapsto (\sigma_1 x_1, \ldots, \sigma_n x_n)$. By [Sch75] there exists a smooth map $\widetilde{f} : \mathbf{R}^n \to \mathbf{R}$ such that $\widetilde{f} \circ s = f \circ s$. As s restricts to a bijection on the underlying sets of $\mathbf{R}^n_+ \to \mathbf{R}^n_+$, the maps f and $\widetilde{f}$ agree on $\mathbf{R}^n_+$, so that f is a restriction of $\widetilde{f}$. □

Corollary 2.1.7. *Let* M *be a smooth manifold with corners, and* N *a smooth manifold* without *corners, then a map* $M \to N$ *is smooth iff there exists a manifold* $\widetilde{M}$ *without corners, an open embedding* $M \subseteq \widetilde{M}$*, and a smooth map* $\widetilde{M} \to N$ *which restricts to* $M \to N$*. In particular, a map* $\Delta^n_{\mathrm{sub}} \to N$ *is smooth iff there exists an open neighbourhood* U *of* Δ^n_{sub} *in* $\mathbf{R}^{n+1}$ *and a smooth map* $U \to N$ *which restricts to* $\Delta^n_{\mathrm{sub}} \to N$*.* □

Example 2.1.8. Consider the unique cocontinuous functor $\widehat{\Delta} \to \mathbf{Diff}^r_{\leq 0}$ carrying $[n]$ to Δ^n_{sub} from Example 2.1.5, then this functor carries the simplicial sets $\partial\Delta^n$ and Λ^n_k to diffeological spaces. These diffeological spaces *are not* equipped with the subspace diffeology of Δ^n_{sub}. Write $\Lambda^2_1 := u_!\Lambda^2_1$ and $\Lambda^2_{1,\mathrm{sub}}$ for the 1-horn of Δ^2 with the subdiffeology. For any path $[0,1] \to \Lambda^2_1$ passing through $\Delta^{\{1\}}$ for some time t_0 there must exist some open neighbourhood U of t_0, which gets constantly mapped to $\Delta^{\{1\}}$. ⌟

2.1.1 Diffeological spaces and descent

A recurring theme in this article is that many ∞-categories consisting of appropriate geometric objects may be profitably studied by embedding them into a suitable ambient ∞-topos. Applying this strategy to diffeological space enables us to recover the main theorem of [Min23] on the classification of diffeological principal bundles (in the sense of [IZ13, 8.11]) as Corollary 2.1.10 below. Thus result is not used in the rest of the article.

Theorem 2.1.9. *Let B be a diffeological space, and G a diffeological group, then for any pullback square*

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d] \\ B \arrow[r] & BG, \end{tikzcd}$$

the map $P \to B$ is a diffeological principal bundle.

Proof. First, we note that $P \to B$ is 0-truncated, as it is the pullback of the 0-truncated map $1 \to BG$. Next, for any plot $\mathbf{R}^d \to B$ the pullback $P|_{\mathbf{R}^d} \to \mathbf{R}^d$ admits local sections and is thus a diffeological principal G-bundle. By faithful descent, the space P is the colimit of all spaces $P|_{\mathbf{R}^d}$. Denote by P' the diffeological space universally associated to P, then by [GL12, §6] the spaces $P|_{\mathbf{R}^d}$ and $P'|_{\mathbf{R}^d}$ are canonically isomorphic, so that P' is likewise the colimit of all spaces $P|_{\mathbf{R}^d}$, and thus isomorphic to P. □

Corollary 2.1.10. *The canonical functor from the groupoid of diffeological principal G-bundles on B to $\mathbf{Diff}^r(B, BG)$ is an equivalence.*

Proof. By [KWW22, Def. 5.1 & Rmk. 5.2] and by descent any diffeological principal G-bundle is classified by a map $B \to BG$. The functor from the groupoid of diffeological principal G-bundles on B to $\mathbf{Diff}^r(B, BG)$ is fully faithful, and by Theorem 2.1.9 it is essentially surjective. □

2.2 Model structures on $\mathbf{Diff}^r$ and related ∞-categories

In this subsection we show that $\mathbf{Diff}^r_{\leq 0}$ is a test category, and construct multiple model structures on $\mathbf{Diff}^r$ and $\mathbf{Diff}^r_{\leq 0}$ with weak equivalences the shape equivalences using the technology of §1.4.2, after which we discuss some of their properties. In §2.2.1 we show that the model structure on $\mathbf{Diff}^r$ transferred using Kihara's simplices restricts to a model structure on $\mathbf{Diff}^r_{\mathrm{concr}}$, which is again Quillen equivalent to $\widehat{\Delta}$, thus recovering a theorem of Kihara. In §2.2.2 we recover Quillen's theorem that the Quillen adjunction $\widehat{\Delta} \rightleftarrows \mathbf{TSpc}$ is a Quillen equivalence by making precise how the model structure on $\mathbf{Diff}^0_{\mathrm{concr}}$ further restricts to $\mathbf{TSpc}$ after "applying a mild homotopy". Furthermore, we sketch how this technique may be used to recover how homotopy colimits may be calculated using the bar construction without prior cofibrant replacement.

Proposition 2.2.1. *The category $\mathbf{Cart}^r$ is a strict test category.*

Proof. By Corollary 1.4.7 it is enough to observe that $\mathbf{R}$ together with the inclusions of $\{0\}$ and $\{1\}$ is a separating interval. □

Theorem 2.2.2 ([Cis03, Th. 6.1.8])**.** *The topos $\mathbf{Diff}^r_{\leq 0}$ is a strict test topos.*

Proof. Combine the preceding proposition with Theorem 1.4.8 and [Clo24b, Cor. 2.3]. □

By Propositions 1.4.13 and 1.1.5 we obtain, respectively, the following two corollaries:

Corollary 2.2.3. *The relative category $\mathbf{Diff}^r_{\leq 0}$ is proper.* □

Corollary 2.2.4. *The following are homotopy colimits in $\mathbf{Diff}^r_{\leq 0}$:*

1. *Pushouts along monomorphisms.*

2. *Filtered colimits.*

3. *Coproducts.*

$\square$

We now discuss several model structures on $\mathbf{Diff}^r$ and $\mathbf{Diff}^r_{\leq 0}$ induced from the nerve functors presented in [Clo24b, §2.2.1], where we have already verified that they satisfy the assumptions of [Clo24b, Th. 1.27], so that they satisfy condition (a) of Proposition 1.4.22 and Theorem 1.4.23. The nerve diagrams are moreover readily seen to satisfy conditions (b) and (c) of Theorem 1.4.23 using Proposition 1.4.24 and Corollary 1.4.26. We thus obtain the following proposition:

Proposition 2.2.5. *The pullback functors along the diagrams*

$\mathbf{A}^\bullet : \Delta \to \mathbf{Diff}^r_{\leq 0}$

$\Delta^\bullet_{\mathrm{sub}} : \Delta \to \mathbf{Diff}^r_{\leq 0}$

$\Delta^\bullet : \Delta \to \mathbf{Diff}^r_{\leq 0}$

$\boldsymbol{\square}^\bullet : \square \to \mathbf{Diff}^r_{\leq 0}$

$\square^\bullet : \square \to \mathbf{Diff}^r_{\leq 0}$

of [Clo24b, §2.2.1] all produce right transferred model structures on $\mathbf{Diff}^r$ *and* $\mathbf{Diff}^r_{\leq 0}$ *in which the weak equivalences are the shape equivalences.* $\square$

Remark 2.2.6. A different proof of Proposition 2.2.5 for the case $\Delta^\bullet_{\mathrm{sub}}$ is given in [Pav22, Th. 7.4]. His argument uses the nerve theorem in a similar way as discussed in [Clo24b, §2.5]. ⌟

For us, the most important of these model structures is the following:

Definition 2.2.7. The model structures on $\mathbf{Diff}^r$ and $\mathbf{Diff}^r_{\leq 0}$ transferred along the pullback functor of the diagram $\Delta^\bullet : \Delta \to \mathbf{Diff}^r_{\leq 0}$ are both called the ***Kihara model structure***, and (trivial) (co)fibrations in this model structure are called ***Kihara (trivial) (co)fibrations***. ⌟

For the convenience of the reader, we repeat the construction of Kihara's simplices: For each $n \geq 1$ and each $0 \leq k \leq n$ we define the set

$$A^n_k := \left\{ \ (x_0, \ldots, x_n) \in \Delta^n \ \middle| \ x_k < 1 \ \right\}.$$

We now proceed inductively: On Δ^0 and Δ^1 the diffeology is the subspace diffeology coming from $\mathbf{R}^1$ and $\mathbf{R}^2$, respectively. Let $n > 1$, and assume that the diffeologies on the simplices Δ^m for $m < n$ have been defined, then we define a diffeology on A^n_k by exhibiting this set as the underlying set of the quotient

$$\begin{tikzcd} \Delta^{n-1} \times \{0\} \arrow[r, hook] \arrow[d, two heads] & \Delta^{n-1} \times [0,1) \arrow[d, two heads] \\ 1 \arrow[r, hook] & A^n_k, \end{tikzcd}$$

where $\Delta^{n-1} \times [0,1) \to A_n^n$ is given by $(x_0, \dots, x_{n-1}; t) \mapsto ((1-t)\cdot x_0, \dots, (1-t)\cdot x_n, t)$, and similarly for $k \neq n$. Finally, the diffeology on Δ^n is determined by the map $\coprod_{k=0}^n A_k^n \twoheadrightarrow \Delta^n$.

Proposition 2.2.8 ([Kih19, § 8])**.** *The horn inclusions $\Lambda_k^n \hookrightarrow \Delta^n$ for $n = 2$ and $n \geq k \geq 0$ admit a deformation retract.* □

Using [Clo24b, Prop. 2.11], we then obtain the following corollary.

Proposition 2.2.9. *All objects in the Kihara model structure are fibrant.* □

From Proposition 1.3.14 we obtain the following corollary:

Corollary 2.2.10 ([BEBP19, Lm 5.10])**.** *The shape functor $\pi_! : \mathbf{Diff}^r \to \mathcal{S}$ commutes with arbitrary products.* □

Assume $r > 0$. We now exhibit a principle which shows that none of the model structures induced from the nerves functors in [Clo24b, §2.2.1] can simultaneously satisfy conditions 1 & 2 discussed in the introduction of this section.

Proposition 2.2.11. *In any model structure on $\mathbf{Diff}^r$ in which the weak equivalences are the shape equivalences, and in which $\{0\} \hookrightarrow \mathbf{R}$ or $\{0\} \hookrightarrow [0,1]$ is a (necessarily trivial) cofibration the following statements cannot* both *be true.*

1. *The model structure is Cartesian.*

2. *All objects are fibrant.*

Proof. We will prove the proposition under the assumption that $\{0\} \hookrightarrow [0,1]$ is a cofibration; the case when $\{0\} \hookrightarrow \mathbf{R}$ is a cofibration is similar. Assume that both 1. & 2. hold. By 1. the pushout product ι of $\Delta^{\{0\}} \hookrightarrow \Delta^1$ with itself is a trivial cofibration, and by 2. all trivial cofibrations admit a retract, which is not true of ι. □

Corollary 2.2.12. *The Kihara model structures on $\mathbf{Diff}^r$ and $\mathbf{Diff}^r_{\leq 0}$ are not Cartesian closed.* □

All the other model structures induced by the nerves in [Clo24b, §2.2.1] are Cartesian closed: The $\mathbf{A}^\bullet$- and $\Delta^\bullet_{\text{sub}}$-model structures by [Pav22, §8], and the $\mathbf{\square}^\bullet$- and $\square^\bullet$-model structures by Propositions A.0.2 & A.0.4.

Corollary 2.2.13. *Not all objects are fibrant in the model structures transferred from the nerves $\mathbf{A}^\bullet$, $\Delta^\bullet_{\text{sub}}$, $\mathbf{\square}^\bullet$, $\square^\bullet$.* □

2.2.1 The Kihara model structure on diffeological spaces

Here we recover Kihara's model structure on diffeological spaces (for $r = \infty$). Moreover, we show that the weak equivalences are the shape equivalences and that the restricted shape functor $\pi_!|_{\mathbf{Diff}^r_{\text{concr}}} : \mathbf{Diff}^r_{\text{concr}} \to \mathcal{S}$ again exhibits $\mathcal{S}$ as the localisation of $\mathbf{Diff}^r_{\text{concr}}$ along the weak equivalences. Finally, we discuss several classes of colimits which are homotopy colimits in $\mathbf{Diff}^r_{\text{concr}}$.

Proposition 2.2.14. *The functor $(\Delta^\bullet)_! : \widehat{\Delta} \to \mathbf{Diff}^r_{\leq 0}$ factors through $\mathbf{Diff}^r_{\text{concr}} \hookrightarrow \mathbf{Diff}^r_{\leq 0}$.*

Proof. The inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ are embeddings, so that all colimits used to construct the realisation of any simplicial set in $\mathbf{Diff}^r_{\leq 0}$ are preserved by the inclusion $\mathbf{Diff}^r_{\mathrm{concr}} \hookrightarrow \mathbf{Diff}^r_{\leq 0}$ by Corollary 1.2.12. $\square$

Remark 2.2.15. Observe that Proposition 2.2.14 fails for the closed simplices Δ^n_{sub}, precisely because the maps $\partial\Delta^n_{\mathrm{sub}} \hookrightarrow \Delta^n_{\mathrm{sub}}$ are not embeddings. See [Pav22, §6] for a proof of this fact. ⌟

Theorem 2.2.16 ([Kih19, Th. 1.3] [Kih17, Th. 1.1])**.** *There exists a cofibrantly generated model structure on* $\mathbf{Diff}^r_{\mathrm{concr}}$*, such that*

(1) *the weak equivalences are the shape equivalences,*

(2) *the generating cofibrations and trivial cofibrations are given by* $\{\partial\Delta^n \hookrightarrow \Delta^n\}_{n\geq 0}$ *and* $\{\Lambda^n_k \hookrightarrow \Delta^n\}_{n\geq 1,\, n\geq k\geq 0}$*,*

(3) *the adjunction* $\widehat{\Delta} \rightleftarrows \mathbf{Diff}^r_{\mathrm{concr}}$ *is a Quillen equivalence, and*

(4) *all objects in* $\mathbf{Diff}^r_{\mathrm{concr}}$ *are fibrant.*

Proof. We shall transfer the model structure from $\widehat{\Delta}$ using Proposition 1.4.19, and make heavy use of (1), which follows from [Clo24b, Prop. 2.13]. Thus, let X be a diffeological space, and consider a map $f : \Lambda^n_k \to X$ ($n \geq 1$, $n \geq k \geq 0$), then $X \to X \cup_f \Delta^n$ is a Δ^1-deformation retract (and thus a weak equivalence), since $\Lambda^n_k \hookrightarrow \Delta^n$ is one. The transfinite composition of $\{\Lambda^n_k \hookrightarrow \Delta^n\}$-cell-attachments is a weak equivalence by Proposition 1.2.12. Lastly, shape equivalences are closed under retract, because isomorphisms are closed under retracts in $\mathcal{S}$.

By Proposition 2.2.14 both adjoints in $\widehat{\Delta} \rightleftarrows \mathbf{Diff}^r_{\mathrm{concr}}$ preserve weak equivalences, and the unit and counit are weak natural equivalences by Theorem 1.4.23, establishing (3).

Finally, (4) follows from the fact that all inclusions $\Lambda^n_k \to \Delta^n$ ($n \geq 1$, $n \geq k \geq 0$) are deformation retracts. $\square$

By Corollary 1.2.12 we obtain the following result.

Proposition 2.2.17. *The following classes of colimits are homotopy colimits in* $\mathbf{Diff}^r_{\mathrm{concr}}$*:*

1. *Pushouts of embeddings along monomorphisms.*

2. *Filtered colimits where all transition morphisms are monomorphisms.*

3. *Arbitrary coproducts*

$\square$

2.2.2 The Quillen model structure on topological spaces

Milnor's result from [Mil57] that the homotopy categories of CW complexes and Kan complexes are equivalent may be seen as the starting point of abstract homotopy theory, as it lays the groundwork for viewing homotopy types as objects of study in their own right. Quillen refined Milnor's result in [Qui67] by showing:

Theorem 2.2.18. *The adjunction*

$$|\_| : \widehat{\Delta} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{TSpc} : s \tag{9}$$

is a Quillen equivalence.

By [Qui67, Lms. 2.3.1 & 2.3.2] the model structure on $\mathbf{TSpc}$ is transferred from $\widehat{\Delta}$, so we are thus in a situation similar to the one encountered for the various cosimplicial diagrams $\Delta \to \mathbf{Diff}^0$ and $\Delta \to \mathbf{Diff}^0_{\mathrm{concr}}$ seen above, and we will give a sketch of how our techniques may be used to give a conceptual proof of why (9) is a Quillen equivalence.

Sketch of proof of Theorem 2.2.18. As in our setup weak equivalences are created by the total singular complex functor, we need to show that the unit is a natural weak equivalence so that we may apply [Hov99, Cor. 1.3.16], i.e., we must recover Milnor's theorem (which we do for all simplicial sets, not just Kan complexes). We denote by $|\_|_{\mathbf{Diff}^0_{\mathrm{concr}}} : \widehat{\Delta} \to \mathbf{Diff}^0_{\mathrm{concr}}$ and $|\_|_{\mathbf{TSpc}} : \widehat{\Delta} \to \mathbf{TSpc}$ the Yoneda extensions along the diagram $\Delta^\bullet_{\mathrm{sub}} : \Delta \to \mathbf{Diff}^0_{\leq 0}$ (see [Clo24b, §2.2.1]), so that $|\_|_{\mathbf{TSpc}}$ is just the usual topological realistion. Observe that the subcategory $\Delta\mathbf{TSpc} \hookrightarrow \mathbf{TSpc}$ spanned by the Δ-generated topological spaces (see [CSW14]) is exhibited as a subcategory of $\mathbf{Diff}^0_{\mathrm{concr}}$ by v^* (see [Clo24b, §2.3.2]), and that $|\_|_{\mathbf{TSpc}}$ factor through $\Delta\mathbf{TSpc}$. As hinted at above, one might then be tempted to implement the same strategy used for constructing a Quillen equivalence between $\widehat{\Delta}$ and $\mathbf{Diff}^0_{\mathrm{concr}}$ in §2.2.1, but, unfortunately, the realisation functor $|\_|_{\mathbf{Diff}^0_{\mathrm{concr}}} : \widehat{\Delta} \to \mathbf{Diff}^0_{\mathrm{concr}}$ does *not* factor through $\Delta\mathbf{TSpc}$. To see this, note for example that the topological space $|\Lambda^2_1|_{\mathbf{TSpc}}$ is homeomorphic to $[0,1]$, but that by Example 2.1.8 the object $|\Lambda^2_1|_{\mathbf{Diff}^0_{\mathrm{concr}}}$ is not even a topological space. Luckily, the realisation in $\mathbf{Diff}^0_{\mathrm{concr}}$ is close enough to the topological realisation for our above strategy to work after a slight modification. All we need to do is show that for any simplicial set X the morphism $|X|_{\mathbf{Diff}^0_{\mathrm{concr}}} \to |X|_{\mathbf{TSpc}}$ is an $\mathbf{R}$-homotopy equivalence. Then, from the commutative diagram

$$\begin{tikzcd} & s|X|_{\mathbf{Diff}^0_{\mathrm{concr}}} \arrow[d] \\ X \arrow[ur] \arrow[r] & s|X|_{\mathbf{TSpc}} \end{tikzcd}$$

we recover Milnor's theorem from the 2-out-of-3 property.

We have just seen that while the map on underlying sets of $|X|_{\mathbf{Diff}^0_{\mathrm{concr}}} \to |X|_{\mathbf{TSpc}}$ is a bijection, it is not true that the map in the other direction is continuous. In order to remedy this, we construct below a homotopy $H^n : [0,1] \times \Delta^n \to \Delta^n$ (in $\Delta\mathbf{TSpc}$) for every $n \geq 1$, which deforms a neighbourhood of $\partial\Delta^n$ down to $\partial\Delta^n$ in such a way that the restriction of H^n to any face Δ^{n-1} yields the homotopy H^{n-1}. For any simplicial set X these homotopies assemble to the two homotopies $H^X : [0,1] \times |X|_{\mathbf{TSpc}} \to |X|_{\mathbf{TSpc}}$ and $H^X : [0,1] \times |X|_{\mathbf{Diff}^0_{\mathrm{concr}}} \to |X|_{\mathbf{Diff}^0_{\mathrm{concr}}}$. The map $|X|_{\mathbf{TSpc}} \xrightarrow{H^X_1} |X|_{\mathbf{Diff}^0_{\mathrm{concr}}}$ is continuous, so that the maps $|X|_{\mathbf{Diff}^0_{\mathrm{concr}}} \to |X|_{\mathbf{TSpc}}$ and $H^X_1 : |X|_{\mathbf{TSpc}} \to |X|_{\mathbf{Diff}^0_{\mathrm{concr}}}$ are then homotopy inverse to each other. <u>Construction of H_n:</u> For each $n \geq 0$ consider the smooth map $\Delta^n \to \mathbf{R}$, $x \mapsto 1 - \|x\|$, i.e., the map that radially measures the distance from any point in Δ^n to the unit sphere in $\mathbf{R}^{n+1}$, and take its gradient, which we view as an electric field. For $t \in [0,1]$ the map $H^n_t : \Delta^n \to \Delta^n$ is then given by viewing a point of Δ^n as a charged particle, which is pushed by the electric field for time t. If the particle hits a face of dimension k, then it is pushed by the component of the electric field parallel to the k-dimensional face, until either $t = 1$ of it hits a face of even lower dimension.

$\square$

Lurie has also recently produced a proof of Milnor's theorem in [Lur22, Tag 0142]. We reframe one of Lurie's key arguments below to obtain yet another proof of Milnor's theorem, this time in the spirit of [Clo24b, §2.3.2]:

Claim: Let K be a finite simplicial set for which $|K|_{\mathbf{Diff}^0_{\mathrm{concr}}} \to |K|_{\mathbf{TSpc}}$ is a weak equivalence, then for any map $f : \partial\Delta^n \to K$ the map $|K \cup_f \Delta^n|_{\mathbf{Diff}^0_{\mathrm{concr}}} \to |K \cup_f \Delta^n|_{\mathbf{TSpc}}$ is likewise a weak equivalence.

From the claim it follows inductively that $|K|_{\mathbf{Diff}^0_{\mathrm{concr}}} \to |K|_{\mathbf{TSpc}}$ is a weak equivalence for all finite simplicial sets K. An arbitrary simplicial set X may then be written as the filtered colimit of its finite simplicial subsets $\{K \subseteq X\}$. By [DI04, Lm. A.3] any map $\mathbf{R}^d \to |X|_{\mathbf{TSpc}}$ factors locally through $|K|_{\mathbf{TSpc}}$ for some finite simplicial subset $K \subseteq X$, so that the colimit of the functor $\{K \subseteq X\} \to \mathbf{Diff}^0_{\mathrm{concr}}$, $K \mapsto |X|_{\mathbf{TSpc}}$, is a topological space. Then for an arbitrary simplicial set X the comparison map $|X|_{\mathbf{Diff}^0_{\mathrm{concr}}} \to |X|_{\mathbf{TSpc}}$ may be written as $\operatorname{colim}_{\{K\subseteq X\}} |K|_{\mathbf{Diff}^0_{\mathrm{concr}}} \to \operatorname{colim}_{\{K\subseteq X\}} |K|_{\mathbf{TSpc}}$ and is thus a weak equivalence by the claim.

Proof of claim: Denote by 0 the centre of Δ^n. As $|\_|_{\mathbf{TSpc}}$ preserves colimits, $K \cup_f \Delta^n$ is sent to $|K|_{\mathbf{TSpc}} \cup_f |\Delta^n|_{\mathbf{TSpc}}$, which can equivalently be written as the pushout

$$\begin{tikzcd}
|\Delta^n|_{\mathbf{TSpc}} \setminus \{0\} \arrow[r] \arrow[d, hook] & |K|_{\mathbf{TSpc}} \cup_f (|\Delta^n|_{\mathbf{TSpc}} \setminus \{0\}) \arrow[d, hook] \\
|\Delta^n|_{\mathbf{TSpc}} \arrow[r] & |K|_{\mathbf{TSpc}} \cup_f |\Delta^n|_{\mathbf{TSpc}}
\end{tikzcd}$$

which is a homotopy pushout by Lurie's Seifert-Van Kampen theorem, [Clo24b, Th. 2.29].

Remark 2.2.19. Let A be a small ordinary category, and $X : A \to \mathbf{TSpc}$ a diagram. Before being proved in [DI04, Th. A.7], it was long a folklore theorem that the topological realisation of the simplicial topological space

$$\cdots \; \substack{\to\\\to\\\to\\\to} \coprod_{a_0\to a_1\to a_2\in A^{\Delta^2}} X_{a_0} \; \substack{\to\\\to\\\to} \coprod_{a_0\to a_1\in A^{\Delta^1}} X_{a_0} \rightrightarrows \coprod_{a_0\in A^{\Delta^0}} X_{a_0} \tag{10}$$

computes the homotopy colimit of X. The sequence $\{H^n\}$ of homotopies used in our proof of Theorem 2.2.18 may also be used to show that the comparison morphism between the realisations of (10) in $\mathbf{TSpc}$ and in $\mathbf{Diff}^\infty$ is an $\mathbf{R}$-homotopy equivalence, thus yielding a new proof of [DI04, Th. A.7]. ⌟

2.3 The differentiable Oka principle

We now implement our strategy for proving Theorem 2.3.28 described in the introduction of this section. In §2.3.1 we construct the squishy fibrations, and use them to exhibit the Kihara boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ as Oka cofibrations. Then, in §2.3.2 we discuss various closure properties — such as being closed under Δ^1-homotopy equivalence — for differentiable sheaves satisfying the differentiable Oka principle. Using the closure properties discussed in §2.3.2 we show in §2.3.3 that simplicial complexes built using Kihara's simplices are Oka cofibrant, and then use an argument originally due to Segal and tom Dieck showing that a large class of (possibly infinite dimensional) differentiable manifolds are Δ^1-homotopy equivalent to such simplicial complexes. Finally, in §2.3.4 we discuss some examples of objects not satisfying the differentiable Oka principle such as the long line.

2.3.1 Squishy fibrations

The squishy fibrations are defined using a cubical diagram of $\boxdot : \square \to \mathrm{Pro}(\mathbf{Diff}^r)$ of *squishy cubes*. To construct these, we first define a precursor, the ε-squishy cubes $\square_\varepsilon : \square \to \mathbf{Diff}^r_{\leq 0}$ for all $0 < \varepsilon < \frac{1}{2}$; these induce the ε-squishy model structures on $\mathbf{Diff}^r$ and $\mathbf{Diff}^r_{\leq 0}$. The squishy cubes are then obtained as the pro-limit of the ε-squishy cubes. We then show that the squishy fibrations are sharp in Proposition 2.3.12, and prove that Kihara's horn inclusions are Oka cofibrations in Theorem 2.3.16.

We will make frequent use of the following ancillary function throughout §2.3.1.

Notation 2.3.1. Let $0 < \alpha < \beta < \frac{1}{2}$, then $\lambda^\beta_\alpha : [0,1] \to [0,1]$ denotes any map such that

(a) $\lambda^\beta_\alpha|_{[0,\alpha]} \equiv 0$, $\lambda^\beta_\alpha|_{[1-\alpha,1]} \equiv 1$,

(b) $\lambda^\beta_\alpha(t) = t$ for all $t \in \left[\frac{1}{2}(\beta+\alpha), 1 - \frac{1}{2}(\beta+\alpha)\right]$, and

(c) $\dot{\lambda}^\beta_\alpha(t) > 0$ for all $t \in (\alpha, 1-\alpha)$.

⌟

ε-squishy intervals and cubes Throughout this subsection fix $0 < \varepsilon < \frac{1}{2}$.

Definition 2.3.2. The pushout of the span

$$\begin{tikzcd} {[0,\varepsilon] \cup [1-\varepsilon,1]} \arrow[r, two heads] \arrow[d, hook] & \{0\} \cup \{1\} \\ \square^1 & \end{tikzcd}$$

(in $\mathbf{Diff}^r$) is called the ***ε-squishy interval*** and is denoted by $\square^1_\varepsilon$. For any $n \in \mathbf{N}$ the n-fold product of $\square^1_\varepsilon$ is called the ***ε-squishy n-cube***, and is denoted by $\square^n_\varepsilon$. ⌟

Proposition 2.3.3. *The ε-squishy n-cube $\square^n_\varepsilon$ is 0-truncated for all $n \in \mathbf{N}$.*

Proof. This is an immediate consequence of Lemma 1.1.1. □

By Proposition A.0.2 we obtain a cocubical diagram

$$\begin{aligned} \square^\bullet_\varepsilon : \quad \square \quad &\to \quad \mathbf{Diff}^r_{\leq 0} \\ \square^n \quad &\mapsto \quad \square^n_\varepsilon. \end{aligned}$$

Notation 2.3.4. We write

$$\begin{array}{rcl} \partial\square^n_\varepsilon & := & (\square^\bullet_\varepsilon)_! \partial\square^n, \quad n \geq 0 \\ \sqcap^n_{k,\xi,\varepsilon} & := & (\square^\bullet_\varepsilon)_! \sqcap^n_{k,\xi}, \quad n \geq 1, \ n \geq k \geq 0, \xi = 0, 1. \end{array}$$

⌟

Proposition 2.3.5. *The ε-squishy cubes generate* $\mathbf{Diff}^r$ *under colimits.*

Proof. For each $d \geq 0$ and for $0 < \varepsilon' < \varepsilon$ the map $\coprod_{x \in \mathbf{R}^d} \square^d_\varepsilon \xrightarrow{\left((\lambda^\varepsilon_{\varepsilon'})^d + x\right)_{x \in \mathbf{R}^d}} \mathbf{R}^d$ is an effective epimorphism, as it is surjective and admits local sections, so the proposition follows from [Lur18, Prop. 20.4.5.1]. □

Lemma 2.3.6. *The differentiable sheaf* $\square^1_\varepsilon$ *is* $\square^1_\varepsilon$*-contractible.*

Proof. Set $\alpha = \varepsilon$, fix any $\alpha < \beta < \frac{1}{2}$, and write $\lambda := \lambda^\beta_\alpha$. Also, define

$$\begin{array}{rrcl} \mu : & \left[\varepsilon, \frac{1}{2}\right] & \to & \left[\varepsilon, \frac{1}{2}\right] \\ & s & \mapsto & \left(\frac{1}{2} - \varepsilon\right) \cdot \lambda\left(\frac{1}{\frac{1}{2} - \varepsilon}(s - \varepsilon)\right) + \varepsilon, \end{array}$$

and

$$\begin{array}{rrcl} \nu : & \left[\frac{1}{2}, 1 - \varepsilon\right] & \to & \left[\frac{1}{2}, 1 - \varepsilon\right] \\ & s & \mapsto & \left(\frac{1}{2} - \varepsilon\right) \cdot \lambda\left(\frac{1}{\frac{1}{2} - \varepsilon}\left(s - \frac{1}{2}\right)\right) + \frac{1}{2}. \end{array}$$

Consider the map

$$\begin{array}{rrcl} H : & [0,1] \times [0,1] & \to & [0,1] \\ & (s,t) & \mapsto & \begin{cases} t & \text{if} \quad 0 \leq s \leq \varepsilon \\ \frac{1}{\frac{1}{2} - \varepsilon}\left((\lambda(t) - t) \cdot \mu(s) + \frac{1}{2} t - \lambda(t) \cdot \varepsilon \right) & \text{if} \quad \varepsilon \leq s \leq \frac{1}{2} \\ \frac{1 - \varepsilon - \nu(s)}{\frac{1}{2} - \varepsilon} \cdot \lambda(t) & \text{if} \quad \frac{1}{2} \leq s \leq 1 - \varepsilon \\ 0 & \text{if} \quad 1 - \varepsilon \leq s \leq 1. \end{cases} \end{array}$$

(Qualitatively, $H|_{[0,\frac{1}{2}] \times [0,1]}$ interpolates between $t \mapsto t$ and λ, and $H|_{[\frac{1}{2},0] \times [0,1]}$ interpolates between λ and $t \mapsto 0$.)

Writing $\square^2_\varepsilon$ as a colimit of

$$(\{0\} \cup \{1\} \twoheadleftarrow [0,\varepsilon] \cup [1-\varepsilon, 1] \hookrightarrow [0,1]) \times (\{0\} \cup \{1\} \twoheadleftarrow [0,\varepsilon] \cup [1-\varepsilon, 1] \hookrightarrow [0,1])$$

we see that we need to check that the induced map $[0,1] \times [0,1] \to \square^1_\varepsilon$ factors as $\square^1_\varepsilon \times [0,1] \to \square^1_\varepsilon$ and $[0,1] \times \square^1_\varepsilon \to \square^1_\varepsilon$, and that moreover $([0,\varepsilon] \cup [1-\varepsilon,1]) \times ([0,\varepsilon] \cup [1-\varepsilon,1]) \to \square^1_\varepsilon$ factors through $(\{0\} \cup \{1\}) \times (\{0\} \cup \{1\}) \to \square^1_\varepsilon$.

The last point is clear, as well as the fact that H factors through $\square^1_\varepsilon \times [0,1] \to [0,1]$. Thus, we are left with showing the second property.

Observe that $H(s,t) = \lambda(t)$ for $s \in \left(\frac{1}{2} - \delta, \frac{1}{2} + \delta\right)$ and some sufficiently small $\delta > 0$. We will check separately that $H|_{[0,\frac{1}{2}+\delta) \times [0,1]}$ and $H|_{(\frac{1}{2}-\delta,1] \times [0,1]}$ factor through $\left[0, \frac{1}{2} + \delta\right) \times \square^1_\varepsilon \to \square^1_\varepsilon$ and $\left(\frac{1}{2} - \delta, 1\right] \times \square^1_\varepsilon \to [0,1]$, respectively. In the first case, $H(s,t) \in [0,\varepsilon) \cup (1-\varepsilon, 1]$ for all values $t \in$

$[0, \varepsilon) \cup (1-\varepsilon, 1]$, so that $H|_{[0, \frac{1}{2}+\delta) \times [0,\varepsilon)}$ and $H|_{[0, \frac{1}{2}+\delta) \times (1-\varepsilon,1]}$ composed with $[0,1] \to \square^1_\varepsilon$ are constant. In the second case, as $\lambda|_{[0,\varepsilon)} \equiv 0$ and $\lambda|_{(1-\varepsilon,1]} \equiv 1$, the map $H|_{(\frac{1}{2}-\delta,1] \times [0,1]}$ is independent of t for all $t \in [0,\varepsilon) \cup (1-\varepsilon, 1]$. $\square$

Just as for the diagrams discussed in [Clo24b, §2.2.1], the diagram $\square \to \mathbf{Diff}^r_{\leq 0}$ induced from $\square^1_\varepsilon$ via Proposition A.0.2 satisfies the assumptions of Proposition 1.4.22 and Theorem 1.4.23, yielding the following proposition:

Proposition 2.3.7. *The pullback functors along the diagrams $\square \to \mathbf{Diff}^r_{\leq 0}$ produces right transferred model structures on $\mathbf{Diff}^r$ and $\mathbf{Diff}^r_{\leq 0}$ in which the weak equivalences are the shape equivalences.* $\square$

Squishy intervals and cubes

Definition 2.3.8. The pro-differentiable sheaf

$$\boxbox^1 := \text{“}\lim_{\varepsilon>0}\text{”}\, \square^1_\varepsilon$$

is called the ***squishy interval***. For any $n \in \mathbf{N}$ the n-fold product of $\boxbox^1$ is called the ***squishy n-cube***, and is denoted by $\boxbox^n$. The resulting cocubical pro-object is denoted by

$$\begin{array}{rccc} \boxbox^\bullet : & \square & \to & \mathrm{Pro}(\mathbf{Diff}^r) \\ & \square^n & \mapsto & \boxbox^n \end{array}$$

⌟

By Proposition C.0.3 the functor $\boxbox^\bullet : \square \to \mathrm{Pro}(\mathbf{Diff}^r)$ may thus be extended to a colimit preserving functor $\boxbox^\bullet_! : [\square^{\mathrm{op}}, \mathcal{S}] \to \mathrm{Pro}(\mathbf{Diff}^r)$.

Notation 2.3.9. We write

$$\begin{array}{rcll} \partial\boxbox^n & := & \boxbox^\bullet_! \partial\square^n, & n \geq 0 \\ \sqcap\!\!\sqcap^n_{k,\xi} & := & \boxbox^\bullet_! \sqcap^n_{k,\xi}, & n \geq 1,\ n \geq k \geq 0, \xi = 0, 1. \end{array}$$

⌟

Proposition 2.3.10. *There is a canonical isomorphism*

$$\boxbox^n \simeq \text{“}\lim_{\varepsilon>0}\text{”}\, \square^n_\varepsilon \quad n \geq 0.$$

Proof. There is an isomorphism $\boxbox^n \simeq \text{“}\lim_{(\varepsilon_1>0)\times\cdots\times(\varepsilon_n>0)}\text{”} \square^1_{\varepsilon_1} \times \cdots \times \square^1_{\varepsilon_n}$ by the proof of Proposition C.0.1. As the ordered set $\left(0, \frac{1}{2}\right)$ admits products it is sifted, and the diagonal map $\left(0, \frac{1}{2}\right) \to \left(0, \frac{1}{2}\right) \times \cdots \times \left(0, \frac{1}{2}\right)$ is initial so that the induced map $\text{“}\lim_{\varepsilon>0}\text{”} \square^1_\varepsilon \times \cdots \times \square^1_\varepsilon \to \text{“}\lim_{(\varepsilon_1>0)\times\cdots\times(\varepsilon_n>0)}\text{”} \square^1_{\varepsilon_1} \times \cdots \times \square^1_{\varepsilon_n}$ is an isomorphism. $\square$

Squishy fibrations

Definition 2.3.11. A morphism $X \to Y$ in $\mathbf{Diff}^r$ is called a ***squishy fibration*** if the morphism of cubical homotopy types $(\boxbox^\bullet)^* X \to (\boxbox^\bullet)^* Y$ is a shape fibration. ⌟

Proposition 2.3.12. *Any squishy fibration is sharp.*

Proof. We will show that the functor satisfies the conditions of Proposition 1.3.13.

The inclusion $\mathrm{Pro}(\mathbf{Diff}^r) \hookleftarrow \mathbf{Diff}^r$ preserves finite limits by [Lur09, Prop. 5.3.5.14], and $[\square^{\mathrm{op}}, \mathcal{S}] \leftarrow \mathrm{Pro}(\mathbf{Diff}^r) : (\square^\bullet)^*$ preserves all limits, as it is a right adjoint.

We conclude with the following two steps, which follow from Proposition 2.3.7 and the fact that shape equivalences are closed under colimits:

Let $X \to Y$ be a shape equivalence in $\mathbf{Diff}^r$ then we have

$$\begin{aligned}
&\underline{\mathbf{Diff}}^r(\square^\bullet, X) \to \underline{\mathbf{Diff}}^r(\square^\bullet, Y)\\
&= \underline{\mathbf{Diff}}^r(\operatorname*{colim}_{\varepsilon>0} \square^\bullet_\varepsilon, X) \to \underline{\mathbf{Diff}}^r(\operatorname*{colim}_{\varepsilon>0} \square^\bullet_\varepsilon, Y)\\
&= \operatorname*{colim}_{\varepsilon>0} \underline{\mathbf{Diff}}^r(\square^\bullet_\varepsilon, X) \to \operatorname*{colim}_{\varepsilon>0} \underline{\mathbf{Diff}}^r(\square^\bullet_\varepsilon, Y)\\
&= \operatorname*{colim}_{\varepsilon>0} \big(\underline{\mathbf{Diff}}^r(\square^\bullet_\varepsilon, X) \to \underline{\mathbf{Diff}}^r(\square^\bullet_\varepsilon, Y)\big).
\end{aligned}$$

Finally, the base change map $\operatorname{colim}_{[n]\in\Delta} \circ (\square^n)^* \to \pi_!$ is a natural isomorphism, as for each differentiable sheaf X we have

$$\begin{aligned}
\operatorname*{colim}_{[n]\in\Delta} \circ (\square^n)^* X &= \operatorname*{colim}_{[n]\in\Delta} \underline{\mathbf{Diff}}^r(\square^n, X)\\
&= \operatorname*{colim}_{[n]\in\Delta} \underline{\mathbf{Diff}}^r(\operatorname*{colim}_{\varepsilon>0} \square^n_\varepsilon, X)\\
&= \operatorname*{colim}_{[n]\in\Delta} \operatorname*{colim}_{\varepsilon>0} \underline{\mathbf{Diff}}^r(\square^n_\varepsilon, X)\\
&= \operatorname*{colim}_{\varepsilon>0} \operatorname*{colim}_{[n]\in\Delta} \underline{\mathbf{Diff}}^r(\square^n_\varepsilon, X)\\
&= \operatorname*{colim}_{\varepsilon>0} \pi_! X\\
&= \pi_! X.
\end{aligned}$$

Finally, the conclusion follows from Proposition 1.4.21. ☐

Remark 2.3.13. It is possible to show that the squishy fibrations together with the shape equivalences form a fibration structure, yielding a different proof that fibrations are sharp. ⌟

Proposition 2.3.14. *The squishy fibrations are closed under arbitrary products.*

Proof. This follows from the fact that the squishy fibrations may be characterised as those morphisms which lift against all horn inclusions $\sqcap^n_{k,\xi} \hookrightarrow \square^n$. ☐

Lemma 2.3.15. *Let $0 < \varepsilon' < \varepsilon < \frac{1}{2}$, then the triangle*

$$\begin{tikzcd} \square^1 \arrow[rr, "\lambda^{\varepsilon}_{\varepsilon'}"] \arrow[dr, two heads] & & \square^1 \arrow[dl, two heads] \\ & \square^1_\varepsilon & \end{tikzcd}$$

commutes.

Proof. It is enough to show that composing $[\varepsilon', 1-\varepsilon'] \to \square^1 \to \square^1_\varepsilon$ yields an epimorphism, then the statement follows from the observation that the triangle

$$\begin{tikzcd} & {[\varepsilon', 1-\varepsilon']} \arrow[dl] \arrow[dr] & \\ \square^1_\varepsilon \arrow[rr, "{[\lambda^{\varepsilon}_{\varepsilon'}]}"] & & \square^1_\varepsilon \end{tikzcd}$$

commutes. To see this, let X be any differentiable space, then any map $f : \square^1 \to X$, which descends to a map $\square^1_\varepsilon \to X$, may be obtained by glueing $f|_{(\varepsilon', 1-\varepsilon')} : (\varepsilon', 1-\varepsilon') \to X$ with $\big[0, \frac{1}{2}(\varepsilon' + \varepsilon)\big) \to 1 \xrightarrow{f(\varepsilon')} X$ and $\big(1 - \frac{1}{2}(\varepsilon' + \varepsilon), 1\big] \to 1 \xrightarrow{f(1-\varepsilon')} X$ along their common intersection. $\square$

Theorem 2.3.16. *Let X be a differentiable sheaf, then*

$$X^{\Delta^n} \to X^{\partial\Delta^n}$$

is a squishy fibration for any $n \geq 0$.

Proof. In this proof we use the following notation $\big(0 < \varepsilon < \frac{1}{2}\big)$:

$$\begin{aligned} \square^k \star_{i,\xi} \Delta^n &:= \Big(\sqcap^k_{i,\xi} \times \Delta^n\Big) \sqcup_{\sqcap^k_{i,\xi} \times \partial\Delta^n} \Big(\square^k \times \partial\Delta^n\Big) \\ \boxbar^k \star_{i,\xi} \Delta^n &:= \Big(\mathbb{\sqcap}^k_{i,\xi} \times \Delta^n\Big) \sqcup_{\mathbb{\sqcap}^k_{i,\xi} \times \partial\Delta^n} \Big(\boxbar^k \times \partial\Delta^n\Big) \\ \square^k_\varepsilon \star_{i,\xi} \Delta^n &:= \Big(\sqcap^k_{i,\xi,\varepsilon} \times \Delta^n\Big) \sqcup_{\sqcap^k_{i,\xi,\varepsilon} \times \partial\Delta^n} \Big(\square^k_\varepsilon \times \partial\Delta^n\Big) \end{aligned}$$

We must show that for every $n \geq 1$, $n \geq k \geq 0$ and $\xi = 0, 1$

$$\begin{tikzcd} \boxbar^k \star_{i,\xi} \Delta^n \arrow[r] \arrow[d, hook] & X \\ \boxbar^n \times \Delta^n & \end{tikzcd} \tag{11}$$

admits a lift. The horizontal map is represented by a map

$$\square^k \star_{i,\xi} \Delta^n \to X$$

which factors through $\square^k_\varepsilon \star_{i,\xi} \Delta^n$ for some $0 < \varepsilon < \frac{1}{2}$. Fix $0 < \varepsilon' < \varepsilon$, and write $\lambda := \lambda_{\varepsilon'}$. To prove the statement we define maps $\mu, \nu : \square^k \times \Delta^n \to \square^k \times \Delta^n$ such that the digram

$$\begin{tikzcd}
\square^k \star_{i,\xi} \Delta^n \arrow[rrrrr, "\left(\lambda^k \times \mathrm{id}_{\Delta^n}|_{\square^k \star_{i,\xi} \Delta^n}\right)^2"] \arrow[d, hook] & & & & & \square^k \star_{i,\xi} \Delta^n \arrow[d, hook] \\
\square^k \times \Delta^n \arrow[r, "\lambda^k \times \mathrm{id}_{\Delta^n}"] & \square^k \times \Delta^n \arrow[r, "\mu"] & \square^k \times \Delta^n \arrow[r, "\nu"] & \square^k \times \Delta^n \arrow[r, "\lambda^k \times \mathrm{id}_{\Delta^n}"] & \square^k \times \Delta^n
\end{tikzcd}$$

commutes and admits a diagonal lift. (Qualitatively, the first instance of $\lambda^k \times \mathrm{id}_{\Delta^n}$ ensures that the resulting lift factors through $\square^k_{\varepsilon'} \times \Delta^n$, μ is a first approximation to the desired retract, next ν completes the retraction in the "Δ^n-direction", and, finally, the second instance of $\lambda^k \times \mathrm{id}_{\Delta^n}$ completes the retract in the "$\square^k$-direction".) Recall, that by Lemma 2.3.15 the map $\lambda^k \times \mathrm{id}_{\Delta^n} : \square^k \times \Delta^n \to \square^k \times \Delta^n$ descends to the identity map $\mathrm{id} : \square^k_\varepsilon \times \Delta^n \to \square^k_\varepsilon \times \Delta^n$, so that the diagram

$$\begin{tikzcd}
\square^k \star_{i,\xi} \Delta^n \arrow[r] \arrow[d] & \square^k \star_{i,\xi} \Delta^n \arrow[r] & \square^k_\varepsilon \star_{i,\xi} \Delta^n \\
\square^k \times \Delta^n \arrow[ur]
\end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd}
\square^k_{\varepsilon'} \star_{i,\xi} \Delta^n \arrow[r] \arrow[d] & \square^k_\varepsilon \star_{i,\xi} \Delta^n \\
\square^k_{\varepsilon'} \times \Delta^n \arrow[ur]
\end{tikzcd}$$

and thus a commutative diagram

$$\begin{tikzcd}
\square^k_{\varepsilon'} \star_{i,\xi} \Delta^n \arrow[r] \arrow[d] & X \\
\square^k_{\varepsilon'} \times \Delta^n \arrow[ur]
\end{tikzcd}$$

which descends to a lift of (11).

<u>Construction of μ and ν:</u> In order to ease the notational burden we will only define μ and ν for $i = k$ and $\xi = 1$.

To define μ, we require an auxiliary smooth function $\rho : \square^{k-1} \times \Delta^n \to \square^1$, such that

(a) $\rho(t_1, \ldots, t_k, s_0, \ldots, s_n) = 1$ if $t_1, \ldots, t_k > \frac{2}{3} \cdot \varepsilon'$ or $s_0 + \cdots + s_n > \frac{2}{3}$;

(b) $\rho(t_1, \ldots, t_k, s_0, \ldots, s_n) = 0$ if $t_1, \ldots, t_k < \frac{1}{3} \cdot \varepsilon'$ and $s_0 + \cdots + s_n < \frac{1}{3}$.

Then, we define

$$\begin{array}{rrcl} \mu : & \square^k \times \Delta^n & \to & \square^k \times \Delta^n \\ & ((t_1, \ldots, t_k), s) & \mapsto & ((t_1, \ldots, t_{k-1}, \rho(t_1, \ldots, t_{k-1}, s) \cdot t_k), s). \end{array}$$

Using partition of unity one can patch together the retractions $\Delta^n \to \Lambda^n_{k_2}$, $1 \leq k_2 \leq n$ to obtain a retract $\sigma : \left\{ (s_0, \ldots, s_n) \in \Delta^n \;\middle|\; s_0 + \cdots + s_n > \frac{1}{3} \right\} \to \partial\Delta^n$. Now, let $\tau : \square^1 \to \square^1$ be a smooth map such that

(a) $\tau(t) = 1$ for $t > \frac{2}{3} \cdot \varepsilon'$, and

(b) $\tau(t) = 0$ for $t < \frac{1}{3} \cdot \varepsilon'$.

Then, we define

$$\begin{array}{rrcl} \nu : & \square^k \times \Delta^n & \to & \square^k \times \Delta^n \\ & ((t_1, \dots, t_k), s) & \mapsto & \big((t_1, \dots, t_k), \mathrm{id}_{\Delta^n} + (\mathrm{id}_{\Delta^n} + \tau(t_k) \cdot (\sigma - \mathrm{id}_{\Delta^n}))(s) \big). \end{array}$$

Proof of smoothness of lift: By construction, it is clear that the lift is smooth at any point which gets mapped to $\square^k \times \Delta^n \setminus (\square^{k-1} \times \{0\}) \times \partial\Delta^n$. Points which get mapped to $(\square^{k-1} \times \{0\}) \times \partial\Delta^n$ admit a neighbourhood which gets mapped to $(\square^{k-1} \times \{0\}) \times \Delta^n$, which concludes the proof. ☐

Remark 2.3.17. The proof of Theorem 2.3.16 does not imply that the maps $\square^k \star_{i,\xi} \Delta^n \hookrightarrow \square^n \times \Delta^n$ admit a retract; only that they lift against all objects in $\mathbf{Diff}^r$. ⌟

2.3.2 Closure properties of Oka cofibrant objects

Oka cofibrant objects are closed under various operations.

Proposition 2.3.18. *The subcategory of* $\mathbf{Diff}^r$ *of Oka cofibrant objects is closed under arbitrary coproducts.*

Proof. For any collection $\{A_i\}_{i \in I}$ of Oka cofibrant objects and any differentiable sheaf X we have

$$\begin{aligned} \pi_! \underline{\mathbf{Diff}}^r \left(\textstyle\coprod_{i \in I} A_i, X\right) &= \pi_! \textstyle\prod_{i \in I} \underline{\mathbf{Diff}}^r (A_i, X) \\ &= \textstyle\prod_{i \in I} \pi_! \underline{\mathbf{Diff}}^r (A_i, X) \\ &= \textstyle\prod_{i \in I} \mathcal{S}(\pi_! A_i, \pi_! X) \\ &= \mathcal{S}\left(\textstyle\coprod_{i \in I} \pi_! A_i, \pi_! X\right) \\ &= \mathcal{S}\left(\pi_! \textstyle\coprod_{i \in I} A_i, \pi_! X\right) \end{aligned}$$

where the second isomorphism follows from Corollary 2.2.10. ☐

Proposition 2.3.19. *Let* $A : \mathbf{N} \to \mathbf{Diff}^r$ *be a diagram such that each object* A_i *is Oka cofibrant, and such that* $A_i \to A_{i+1}$ *is a cofibration in the Kihara model structure for all* $i \in \mathbf{N}$*, then* $\operatorname{colim} A$ *is Oka cofibrant.*

Proof. Let X be any differentiable sheaf, then

$$\begin{aligned} \pi_! \underline{\mathbf{Diff}}^r (\operatorname{colim} A, X) &= \pi_! \lim \underline{\mathbf{Diff}}^r (A, X) \\ &= \operatorname{colim} (\square^\bullet)^* (\lim \underline{\mathbf{Diff}}^r (A, X)) \\ &= \operatorname{colim} \lim (\square^\bullet)^* (\underline{\mathbf{Diff}}^r (A, X)) \\ &= \lim \operatorname{colim} (\square^\bullet)^* (\underline{\mathbf{Diff}}^r (A, X)) \\ &= \lim \pi_! \underline{\mathbf{Diff}}^r (A, X) \\ &= \lim \mathcal{S}(\pi_! A, \pi_! X) \\ &= \mathcal{S}(\operatorname{colim} \pi_! A, \pi_! X) \\ &= \mathcal{S}(\pi_! \operatorname{colim} A, \pi_! X), \end{aligned}$$

where the fourth isomorphism follows from [MG21, Prop. 1.23]. ∎

Proposition 2.3.20. *The subcategory* $\mathbf{Diff}^r$ *of Oka cofibrant objects is closed under finite products.*

Proof. Let A, B be Oka cofibrant objects, and let X be any differentiable sheaf, then one obtains the following series of canonical equivalences:

$$\begin{aligned}
\pi_! \underline{\mathbf{Diff}}^r(A \times B, X) &= \pi_! \underline{\mathbf{Diff}}^r(A, \underline{\mathbf{Diff}}^r(B, X)) \\
&= \mathcal{S}(\pi_! A, \pi_! \underline{\mathbf{Diff}}^r(B, X)) \\
&= \mathcal{S}(\pi_! A, \mathcal{S}(\pi_! B, \pi_! X)) \\
&= \mathcal{S}(\pi_! A \times \pi_! B, \pi_! X) \\
&= \mathcal{S}(\pi_!(A \times B), \pi_! X).
\end{aligned}$$

∎

Lemma 2.3.21. *The map* $X \to \underline{\mathbf{Diff}}^r(\Delta^1, X)$ *is a* Δ^1*-homotopy equivalence for every object* X *in* $\mathbf{Diff}^r$*.*

Proof. The Δ^1-homotopy inverse is constructed using the inclusion $\Delta^{\{0\}} \hookrightarrow \Delta^1$. The morphisms $\underline{\mathbf{Diff}}^\infty(\Delta^{\{0\}}, X) \to \underline{\mathbf{Diff}}^\infty(\Delta^1, X) \to \underline{\mathbf{Diff}}^\infty(\Delta^{\{0\}}, X)$ compose to the identity, so we are left with showing that the composition of $\underline{\mathbf{Diff}}^\infty(\Delta^1, X) \to \underline{\mathbf{Diff}}^\infty(\Delta^{\{0\}}, X) \to \underline{\mathbf{Diff}}^\infty(\Delta^1, X)$ is Δ^1-homotopic to the identity.

Such a homotopy is given by the transpose of the diagram

$$\begin{tikzcd}
\Delta^{\{0\}} \times \Delta^1 \times X^{\Delta^1} \arrow[r] \arrow[dr] & \Delta^1 \times X^{\Delta^{\{0\}}} \arrow[drr, bend left] & & \\
& \Delta^1 \times \Delta^1 \times X^{\Delta^1} \arrow[r, "\zeta \times \mathrm{id}"] & \Delta^1 \times X^{\Delta^1} \arrow[r, "\mathrm{ev}"] & X \\
\Delta^{\{1\}} \times \Delta^1 \times X^{\Delta^1} \arrow[ur] \arrow[urrr, bend right, "\mathrm{ev}"'] & & &
\end{tikzcd}$$

where $\zeta : \Delta^1 \times \Delta^1 \to \Delta^1$ is given by $(s,t) \mapsto s \cdot t$ (here we identify Δ^1 with $[0,1]$), so we are left with showing that it commutes. The bottom part commutes because the morphisms $\Delta^{\{1\}} \times \Delta^1 \times X^{\Delta^1} \to \Delta^1 \times \Delta^1 \times X^{\Delta^1} \xrightarrow{\zeta \times \mathrm{id}} \Delta^1 \times X^{\Delta^1}$ compose to the identity. The composition of $\Delta^{\{0\}} \times \Delta^1 \times X^{\Delta^1} \to \Delta^1 \times \Delta^1 \times X^{\Delta^1} \xrightarrow{\zeta \times \mathrm{id}} \Delta^1 \times X^{\Delta^1}$ is equivalent to the composition of $\Delta^1 \times X^{\Delta^1} \to \Delta^1 \times X^{\Delta^{\{0\}}} \to \Delta^1 \times X^{\Delta^1}$, so that we are left with proving that the square obtained after composing with the evaluation in

$$\begin{tikzcd}
& \Delta^1 \times X^{\Delta^{\{0\}}} \arrow[dr] & & \\
\Delta^1 \times X^{\Delta^1} \arrow[ur] \arrow[dr] & & \Delta^1 \times X^{\Delta^1} \arrow[r, "\mathrm{ev}"] & X \\
& \Delta^{\{0\}} \times X^{\Delta^1} \arrow[ur] & &
\end{tikzcd}$$

commutes, but this follows from the more general observation that for any map $A \to B$ the square

$$\begin{tikzcd} & B \times X^B \arrow[dr] & \\ A \times X^B \arrow[ur] \arrow[dr] & & X \\ & A \times X^A \arrow[ur] & \end{tikzcd}$$

commutes, which is true because both the top and the bottom composition transpose to the map $X^B \to X^A$; the top one from the general formula of obtaining a transpose using the counit, and the bottom composition by naturality. $\square$

Proposition 2.3.22. *The ∞-category of Oka cofibrant objects is closed under Δ^1-homotopy equivalence.*

Proof. Let A be Oka cofibrant, and $A \to B$ a Δ^1-homotopy equivalence, then we obtain a commutative diagram

$$\begin{tikzcd} \pi_! \underline{\mathbf{Diff}}^r(A, X) \arrow[r] & \mathcal{S}(\pi_! A, \pi_! X) \\ \pi_! \underline{\mathbf{Diff}}^r(B, X) \arrow[r] \arrow[u] & \mathcal{S}(\pi_! B, \pi_! X) \arrow[u] \end{tikzcd}$$

in which we must show that the vertical arrows are isomorphisms, which in turn follows from showing that the functors $\mathcal{S}(\pi_! \_, \pi_! X)$ and $\pi_! \underline{\mathbf{Diff}}^r(\_, X)$ send Δ^1-homotopic maps to equivalent maps. For $\mathcal{S}(\pi_! \_, \pi_! X)$ this is clear, as $\pi_!$ preserves products and $\pi_! \Delta^1 = \mathbf{1}$, for $\pi_! \underline{\mathbf{Diff}}^r(\_, X)$ this follows from Lemma 2.3.21. $\square$

Remark 2.3.23. Proposition 2.3.22 remains true for other intervals than Δ^1, e.g., $\mathbf{R}$ or $\square^1_\varepsilon$ for $0 < \varepsilon < \frac{1}{2}$. ⌟

2.3.3 Proof of the differentiable Oka principle

Throughout §2.3.3 we fix $r = \infty$, as we cite [Kih20, Prop. 9.5 & Th. 11.20] (see Lemma 2.3.26 and Theorem 2.3.28) which are both stated in the smooth setting.

Remark 2.3.24. We are confident that [Kih20, Prop. 9.5] also holds for $r < \infty$, but the classes of manifolds in [Kih20, Th. 11.20] would probably need to be modified, the theory of infinite dimensional manifolds is sensitive to changes in regularity. ⌟

Let X be a diffeological space, and let $U = \{U_\alpha\}_{\alpha \in A}$ be a cover of X, then there exists a $\mathbf{Diff}^\infty_{\mathrm{concr}}$-enriched category X_U with

$$\begin{aligned} \operatorname{Obj} X_U &= \textstyle\coprod_\sigma U_\sigma \\ \operatorname{Mor} X_U &= \textstyle\coprod_{\sigma \supseteq \tau} U_\sigma \end{aligned}$$

where σ, τ denote non-empty finite subsets of A such that $U_\sigma := \bigcap_{\alpha \in \sigma} U_\alpha \neq \varnothing$. The topological realisation of (the nerve of) X_U is denoted by BX_U. The space BX_U may be constructed in stages using the pushouts

$$\begin{tikzcd} \coprod\limits_{\sigma_n \supsetneq \cdots \supsetneq \sigma_0} U_{\sigma_n} \times \partial\Delta^n \arrow[r, hook] \arrow[d, hook] & BX_U^{(n-1)} \arrow[d, hook] \\ \coprod\limits_{\sigma_n \supsetneq \cdots \supsetneq \sigma_0} U_{\sigma_n} \times \Delta^n \arrow[r, hook] & BX_U^{(n)} \end{tikzcd} \tag{12}$$

At each stage one can construct inductively an obvious commutative square obtained by replacing $BX_U^{(n)}$ by X in (12), thus producing a canonical map $BX_U \to X$. As the pushouts at each step satisfy the conditions of Proposition 1.2.8, each stage $BX_U^{(n)}$ is a diffeological space; the object BX_U is then a diffeological space by Proposition 1.2.10, as it is a filtered colimit of diffeological spaces along monomorphisms.

Definition 2.3.25. A covering on a diffeological space is called ***numerable*** if it admits a subordinate partition of unity. ⌟

The original formulation of the following lemma in the setting of topological spaces is due to Segal [Seg68, §4] and tom Dieck [tD71, Th. 4]. Translating these results into the smooth setting is very technical, and is carried out by Kihara in [Kih20, §9].

Lemma 2.3.26 ([Kih20, Prop. 9.5])**.** *Let X be a diffeological space, and let U be a numerable cover of X, then the canonical map $BX_U \to X$ is a Δ^1-homotopy equivalence.* □

Theorem 2.3.27. *Let X be a diffeological space, and let U be a numerable cover of X. If each member of U is Oka cofibrant, then so is X.*

Proof. By Lemma 2.3.26 and Proposition 2.3.22 the space X is Oka cofibrant iff BX_U is. We will show that each stage $BX_U^{(n)}$ is Oka cofibrant, and then conclude that BX_U is Oka cofibrant by Proposition 2.3.19. The diffeological space $BX_U^{(0)}$ is Oka cofibrant by Proposition 2.3.18. Applying $\underline{\mathbf{Diff}}^\infty(\_, X)$ to the square (12) yields the pullback

$$\begin{tikzcd} \underline{\mathbf{Diff}}^\infty(BX_U^{(n)}, X) \arrow[r] \arrow[d] & \prod\limits_{\sigma_n \supsetneq \cdots \supsetneq \sigma_0} \underline{\mathbf{Diff}}^\infty(U_{\sigma_n}, X)^{\Delta^n} \arrow[d] \\ \underline{\mathbf{Diff}}^\infty(BX_U^{(n-1)}, X) \arrow[r] & \prod\limits_{\sigma_n \supsetneq \cdots \supsetneq \sigma_0} \underline{\mathbf{Diff}}^\infty(U_{\sigma_n}, X)^{\partial\Delta^n} \end{tikzcd}$$

in which the vertical morphism to the right is sharp as it is a squishy fibration by Theorem 2.3.16 and Proposition 2.3.14. □

For finite dimensional Hausdorff 2nd countable manifolds the following theorem was first proved in [BEBP19].

Theorem 2.3.28. *Any paracompact Hausdorff C^∞-manifold locally modelled on Hilbert spaces, nuclear Fréchet spaces, or nuclear Silva spaces satisfies the differentiable Oka principle.* □

Proof. The content of [Kih20, Th. 11.20] is precisely that the manifolds considered in the theorem are diffeological spaces satisfying the condition in Theorem 2.3.27. □

The infinite dimensional manifolds considered in Theorem 2.3.28 include many interesting examples, such as the $\underline{\mathbf{Diff}}^{\infty}(M,N)$ or the manifold of submanifolds of N diffeomorphic to M, for M,N smooth finite dimensional paracompact Hausdorff manifolds without corners and M compact.

2.3.4 Counterexamples

There are many directions in which it is not possible to extend Corollary 2.3.28.

Example 2.3.29. $B\mathbf{Z} = \pi_!\underline{\mathbf{Diff}}^r(1,S^1) = \pi_!\underline{\mathbf{Diff}}^r(\pi^*B\mathbf{Z},S^1) \neq \mathcal{S}(B\mathbf{Z},B\mathbf{Z}) = \mathbf{Z}$. ⌟

One must be careful when dropping the Hausdorfness requirement:

Example 2.3.30. Denote by $\mathbf{R}_{\bullet\bullet}$ the real line with two origins, then

$$\begin{aligned} B\mathbf{Z} &= \pi_!\underline{\mathbf{Diff}}^r(\mathbf{R},S^1) \\ &= \pi_!\underline{\mathbf{Diff}}^r(\mathbf{R}_{\bullet\bullet},S^1) \\ &\neq \mathcal{S}(\pi_!\mathbf{R}_{\bullet\bullet},\pi_!S^1) \\ &= \mathcal{S}(B\mathbf{Z},B\mathbf{Z}) \\ &= \mathbf{Z}. \end{aligned}$$

⌟

Example 2.3.31. Denote by $\mathbf{R}_{||}$ the space obtained by glueing two copies of $\mathbf{R}$ along the subspace $(-\infty,-1)\cup(1,\infty)$, then $\mathbf{R}_{||}$ is $\mathbf{R}$-homotopy equivalent to S^1, so that it is Oka cofibrant. In particular,

$$\pi_!\underline{\mathbf{Diff}}^r(\mathbf{R}_{||},S^1) = \pi_!\underline{\mathbf{Diff}}^r(S^1,S^1) = \mathcal{S}(\pi_!S^1,\pi_!S^1) = \mathcal{S}(\pi_!\mathbf{R}_{||},\pi_!S^1).$$

⌟

Non-paracompact manifolds may not be Oka cofibrant:

Example 2.3.32. Let $\mathbf{L}$ denote the long line. It has trivial shape but is not contractible. Thus $\mathcal{S}(\pi_!\mathbf{L},\pi_!\mathbf{L}) = \mathcal{S}(1,1) = 1$, while $\underline{\mathbf{Diff}}^r(\mathbf{L},\mathbf{L})$ has at least two path components. ⌟

Appendix

A The cube category

Here we collect some background material on the cube category for the convenience of the reader.

Definition A.0.1. The ***cube category*** $\square$ is the subcategory of **Set** whose objects are given by $\{0,1\}^n$ for $n \geq 0$, and whose morphisms are generated by the maps

$$\begin{array}{rrcl} \delta_i^\xi : & \square^{n-1} & \to & \square^n \\ & (x_1,\ldots,x_{n-1}) & \mapsto & (x_1,\ldots,x_{i-1},\xi,x_i,\ldots,x_{n-1}) \end{array}$$

for $n \geq i \geq 1$ and $\varepsilon = 0,1$, and

$$\begin{array}{rrcl} \sigma_i : & \square^{n+1} & \to & \square^n \\ & (x_1,\ldots,x_{n+1}) & \mapsto & (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{n+1}) \end{array}$$

for $n \geq 0$ and $n \geq i \geq 1$. The category of ***cubical sets*** is the category $\widehat{\square}$ of presheaves on $\square$. ⌟

The cube category $\square$ admits a (strict) monoidal structure given by $(\square^m,\square^n) \mapsto \square^{m+n}$ which extends to cubical sets via Day convolution. This monoidal structure is denoted by $\otimes$.

We denote by $\square^{\leq 1}$ the full subcategory of $\square$ spanned by $\square^0,\square^1$.

Proposition A.0.2 ([Cis06, Prop. 8.4.6])**.** *Let M be a monoidal category, then the restriction functor*

$$[\square, M] \to [\square^{\leq 1}, M]$$

induces an equivalence of categories between the full subcategory of $[\square, M]$ spanned by monoidal functors, and the full subcategory of $[\square^{\leq 1}, M]$ spanned by functors sending $\square^0$ to the monoidal unit of M. $\square$

Definition A.0.3. For every $n \geq 0$ the ***boundary of*** $\square^n$ is the subobject $\partial\square^n := \cup_{(j,\zeta)}\mathrm{Im}_{\delta_j^\zeta} \subset \square^n$, and for every $n \geq i \geq 1$ and $\xi = 0,1$ the (i,ξ)***-th horn of*** $\square^n$ is the subobject $\sqcap^n_{i,\xi} := \cup_{(j,\zeta)\neq(i,\xi)}\mathrm{Im}_{\delta_j^\zeta} \subset \square^n$. ⌟

Proposition A.0.4 ([Cis06, Lm. 8.4.36])**.** *For $m \geq 1$, $n \geq k \geq 1$ and $\varepsilon = 0,1$ the universal morphisms determined by the pushouts of the spans contained in the commutative squares*

$$\begin{tikzcd} \sqcap^n_{i,\varepsilon} \otimes \partial\square^m \arrow[r] \arrow[d] & \square^n \otimes \partial\square^m \arrow[d] \\ \sqcap^n_{i,\varepsilon} \otimes \square^m \arrow[r] & \square^n \otimes \square^m \end{tikzcd} \qquad \begin{tikzcd} \partial\square^m \otimes \sqcap^n_{i,\varepsilon} \arrow[r] \arrow[d] & \partial\square^n \otimes \square^m \arrow[d] \\ \square^m \otimes \sqcap^n_{i,\varepsilon} \arrow[r] & \square^n \otimes \square^m \end{tikzcd}$$

recover the canonical inclusions $\sqcap^{n+m}_{i,\varepsilon} \hookrightarrow \square^{n+m}$ and $\sqcap^{n+m}_{i+m,\varepsilon} \hookrightarrow \square^{n+m}$ and the universal morphism *determined by the pushout of the span contained in the commutative square*

$$\begin{tikzcd} \partial\square^m \otimes \partial\square^n \arrow[r] \arrow[d] & \partial\square^m \otimes \square^n \arrow[d] \\ \square^m \otimes \partial\square^n \arrow[r] & \square^m \otimes \square^n \end{tikzcd}$$

recovers the inclusion $\partial\square^{m+n} \hookrightarrow \square^{m+n}$. □

Theorem A.0.5 ([Cis06, Cor. 8.4.13 or Prop. 8.4.27])**.** *The cube category* $\square$ *is a test category.* □

Theorem A.0.6 ([Cis06, Th. 8.4.38])**.** *The maps*

(i) $\partial\square^n \hookrightarrow \square^n$ *(*$n \geq 0$*), and*

(ii) $\sqcap^n_{i,\varepsilon} \hookrightarrow \square^n$ *(*$n \geq i \geq 1$, $\varepsilon = 0, 1$*)*

generate, respectively, the cofibrations and acyclic cofibrations of the test model structure on $\widehat{\square}$. □

B Model structures on ∞-categories

Here we collect some basic definitions and facts about model ∞-categories. The proofs are the same as in the ordinary categorical case. Typically, when working in ∞-categories, the coherence of various constructions is ensured via a consistent use of universal properties. However, as the lifting conditions of model structures are not unique such techniques no longer work, and one is forced to perform diagramatic arguments similar to those employed for ordinary categories. We hope that this appendix may serve as an illustration of how such arguments may still be carried out for sufficiently small diagrams in ∞-categories. For safety, we work explicitly with quasi-categories, however, we hasten to point out that all we are really using is that inner anodyne extensions are sent to equivalences of ∞-categories by $\widehat{\Delta} \hookrightarrow [\Delta^{\mathrm{op}}, \mathcal{S}] \xrightarrow{L} \mathbf{Cat}$, where L denotes the left adjoint to the inclusion $[\Delta^{\mathrm{op}}, \mathcal{S}] \hookleftarrow \mathbf{Cat}$.

Definition B.0.1. Let C be an ∞-category, and let $f : a \to b$, $g : x \to y$ be morphisms in C, then f has the ***left lifting property*** w.r.t. g, and g has the ***right lifting property*** w.r.t. f if every commutative square

$$\begin{tikzcd} a \arrow[r] \arrow[d, "f"'] & x \arrow[d, "g"] \\ b \arrow[r] & y \end{tikzcd}$$

may be extended to a 3-simplex

$$\begin{tikzcd} a \arrow[r] \arrow[d, "f"'] & x \arrow[d, "g"] \\ b \arrow[r] \arrow[ur, dashed] & y \end{tikzcd}$$

In this case we write $f \boxslash g$. More generally, if L, R are two collections of morphisms in C, we write $L \boxslash R$ if $f \boxslash g$ for all $f \in L$ and $g \in R$. Finally, we write $L^{\boxslash}$ for the collection of morphism g such that $f \boxslash g$ for all $f \in L$, and ${}^{\boxslash}R$ for the collections of morphisms g such that $f \boxslash g$ for all $g \in R$. ⌟

Definition B.0.2. Let C be an ∞-category, then a pair (L, R) of collections of morphisms in C form a ***weak factorisation system*** if

(a) $L = {}^{\boxslash}R$,

(b) $L^{\boxslash} = R$, and

(c) any morphism $a \to x$ may be factored as

$$\begin{tikzcd} & q \arrow[dr] & \\ a \arrow[ur] \arrow[rr] & & x \end{tikzcd}$$

with $a \to q \in L$ and $q \to x \in R$.

⌟

Remark B.0.3. Note, that we do not require the factorisation of $a \to x$ in Definition B.0.2 to be either unique nor functorial. ⌟

Remark B.0.4. Observe that both classes of a weak factorisation system contain all isomorphisms and are closed under composition (and therefore also under homotopy). ⌟

Proposition B.0.5 ([DAG X, Prop. 1.4.9]). *Let C be an ∞-category, and (L, R) a weak factorisation system, then both classes are closed under retracts, and the left class is closed under transfinite compositions of pushouts of morphisms in L.* □

Proposition B.0.6. *Let C, D be ∞-categories equipped with weak factorisation systems (L, R) and (L', R') respectively, then for any adjunction $f : C \rightleftarrows D : g$ we have*

$$fL \subseteq L' \iff R \supseteq gR'.$$

Proof. Let $a \to b$ and $x \to y$ be morphisms in C and D, respectively, then we we want to show that the transpose of the lift in any square

$$\begin{tikzcd} a \arrow[r] \arrow[d] & gx \arrow[d] \\ b \arrow[r] \arrow[ur, dashed] & gy \end{tikzcd} \tag{13}$$

gives a lift

$$\begin{tikzcd} fa \arrow[r] \arrow[d] & x \arrow[d] \\ fb \arrow[r] \arrow[ur, dashed] & y \end{tikzcd} \tag{14}$$

and vice versa. In the ∞-categorical setting this requires a little bit of care, because lifts of squares correspond to extensions

$$\begin{tikzcd} \Delta^{\{0,1,3\}\cup\{0,2,3\}} \arrow[r] \arrow[d, hook] & C, D \\ \Delta^3 \arrow[ur, dashed] & \end{tikzcd}$$

which must be transported back and forth between C and D. We use some elementary facts about the Joyal model structure and the calculus of simplices to accomplish this. Recall that the datum of exhibiting f and g as adjoint is given by a weak equivalence $A \xrightarrow{\sim} B$ in the Joyal model structure

$$
\begin{tikzcd}
C^{\Delta^1} \arrow[d] & A \arrow[l] \arrow[rr, "\sim"] \arrow[dr, two heads] \arrow[dl, phantom, very near start, "\llcorner"] & & B \arrow[r] \arrow[dl, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & D^{\Delta^1} \arrow[d] \\
C^{\partial\Delta^1} & & C \times D \arrow[ll, "{(\mathrm{id},g)}"'] \arrow[rr, "{(f,\mathrm{id})}"] & & D^{\partial\Delta^1}
\end{tikzcd}
$$

Recalling that $\Delta^{\{0,1,3\}} \cup \Delta^{\{0,2,3\}} \simeq \Delta^1 \times \Delta^1$, exponentiating the above diagram by Δ^1 yields the lower half of the following diagram:

$$
\begin{tikzcd}
C^{\Delta^3} \arrow[d, two heads] & A' \arrow[l] \arrow[d, two heads] \arrow[rr, "\sim"] \arrow[dl, phantom, very near start, "\llcorner"] & & B' \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & D^{\Delta^3} \arrow[d, two heads] \\
C^{\Delta^{\{0,1,3\}} \cup \Delta^{\{0,2,3\}}} \arrow[d, two heads] & A^{\Delta^1} \arrow[l] \arrow[rr, "\sim"] \arrow[dr, two heads] \arrow[dl, phantom, very near start, "\llcorner"] & & B^{\Delta^1} \arrow[r] \arrow[dl, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & D^{\Delta^{\{0,1,3\}} \cup \Delta^{\{0,2,3\}}} \arrow[d] \\
C^{\Delta^{\{0,1\}} \cup \Delta^{\{2,3\}}} & & C^{\Delta^{\{0,1\}}} \times D^{\Delta^{\{2,3\}}} \arrow[ll, "{(\mathrm{id},g)}"'] \arrow[rr, "{(f,\mathrm{id})}"] & & D^{\Delta^{\{0,1\}} \cup \Delta^{\{2,3\}}}
\end{tikzcd}
$$

On fibres over $((a \to b), (x \to y))$ the weak equivalence $A^{\Delta^1} \xrightarrow{\sim} B^{\Delta^1}$ yields an equivalence between the spaces of squares of the form (13) and (14). On fibres over equivalent squares in the spaces $A^{\Delta^1}|_{((a\to b),(x\to y))} \sim B^{\Delta^1}|_{((a\to b),(x\to y))}$ the weak equivalence $A' \xrightarrow{\sim} B'$ yields an equivalence between the respective spaces of lifts. □

Lemma B.0.7 (Retract argument)**.** *Let C be an ∞-category with two sets of maps L, R such that $L^{\boxslash} \supseteq R$ ($L \subseteq {}^{\boxslash}R$). Assume that every morphisms in C factors as a morphism in L followed by a morphism in R, and that R (L) is closed under retracts, then $L^{\boxslash} = R$ ($L = {}^{\boxslash}R$).*

Proof. We will prove that if R is closed under retracts, then $L^{\boxslash} = R$. The other statement is dual. Assume that $x \to y$ is in $L^{\boxslash}$, then we may factor it into a morphism $x \to z$ in L followed by a morphism $z \to y$ and consider the diagram

$$
\begin{tikzcd}
x \arrow[r, "\mathrm{id}"] \arrow[d] & x \arrow[d] \\
z \arrow[r] & y
\end{tikzcd}
$$

which admits a lift by assumption, yielding the retract

$$
\begin{tikzcd}
x \arrow[r] \arrow[d] & z \arrow[r] \arrow[d] & x \arrow[d] \\
y \arrow[r, "\mathrm{id}"] & y \arrow[r, "\mathrm{id}"] & y
\end{tikzcd}
$$

□

Definition B.0.8. Let (M, W) be a relative ∞-category with finite limits and colimits, in which W satisfies the 2-out-of-3 property. A ***model structure*** on M is a pair (C, F) of collections of morphisms in M such that $(C \cap W, F)$ and $(C, F \cap W)$ form weak factorisation systems. A relative ∞-category with finite limits and colimits equipped with a model structure is called a ***model ∞-category***. The morphisms in C (resp. $C \cap W$) are called ***(trivial) cofibrations***, and the morphisms in F (resp. $F \cap W$) are called ***(trivial) fibrations***. ⌟

We give two equivalent characterisations of model structures.

Proposition B.0.9. *Let (M, W) be a relative ∞-category with finite limits and colimits, in which W satisfies the 2-out-of-3 property, then a pair (C, F) of collections of morphisms in M is a model structure iff*

(a) *W, C, F are closed under retracts,*

(b) *$(W \cap C) \boxslash F$, $C \boxslash (W \cap F)$, and*

(c) *any morphism in M factors both as a morphism in $W \cap C$ followed by a morphism in F, as well as a morphism in C followed by a morphism in $W \cap F$.*

Proof. The proof works exactly the same as in the ordinary categorical case, except that we need to keep track of composition data.

We first prove that if the pair (C, F) satisfies the axioms of Definition B.0.8, then it satisfies properties (a) - (c). The classes C and F are closed under retracts by Proposition B.0.5, and (b) & (c) follow by definition. Thus, we are left with showing that W is closed under retracts. To this end we will compile the proof of [JT07, Prop. 7.8] in the ∞-categorical setting (we advise the reader to consult [JT07, Prop. 7.8] if they are not already familiar with the argument). The gap map of the pushout product of $\partial\Delta^1 \hookrightarrow \Delta^1$ and $\Lambda^2_1 \hookrightarrow \Delta^2$ is inner anodyne (see [Cis19, Cor. 3.2.4]), so that the datum of a retract is determined up to contractible choice by a diagram

$$\begin{tikzcd}
\bullet \arrow[r, "\ell"] \arrow[d, "f"'] \arrow[dr] \arrow[rr, bend left, "\mathrm{id}"] & \bullet \arrow[r] \arrow[d, "w"'] \arrow[dr] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & x \arrow[r, "r"'] & y.
\end{tikzcd} \tag{15}$$

We assume that w is in W, and want to show that f is likewise in W. At first, we also assume that f is a fibration, and then we deduce the general case from this special case. We begin by factoring w into a trivial cofibration u followed by a fibration v. This corresponds to glueing the 2-simplex

$$\begin{tikzcd}
\bullet \arrow[dd, "w"'] \arrow[dr, "u"] & \\
& \bullet \arrow[dl, "v"] \\
x &
\end{tikzcd}$$

to the above diagram, i.e., we obtain the diagram

$$\begin{tikzcd}
\bullet \arrow[r, "\ell"] \arrow[dd, "f"'] \arrow[ddr] \arrow[rrr, bend left, "\mathrm{id}"] & \bullet \arrow[rr] \arrow[dd, "w"'] \arrow[dr, "u"] \arrow[ddrr] & & \bullet \arrow[dd, "f"] \\
& & \bullet \arrow[dl, "v"] & \\
\bullet \arrow[r] \arrow[rrr, bend right, "\mathrm{id}"'] & x \arrow[rr, "r"'] & & y.
\end{tikzcd}$$

As the inclusions $\Delta^{\{0,1,3\}\cup\{1,2,3\}} \hookrightarrow \Delta^3$ and $\Delta^{\{0,1,2\}\cup\{0,2,3\}} \hookrightarrow \Delta^3$ are inner anodyne, we may extend the above diagram to an equivalent one, containing two new 3-simplices as indicated,

$$
\begin{tikzcd}
\bullet \arrow[rr, "\ell"] \arrow[rrrr, bend left, "\mathrm{id}"] \arrow[dd, "f"'] \arrow[ddrr] \arrow[drrr] & & \bullet \arrow[rr] \arrow[dd, "w"'] \arrow[dr, "u"] \arrow[ddrr] & & \bullet \arrow[dd, "f"] \\
& & & \bullet \arrow[dl, "v"] \arrow[dr] & \\
\bullet \arrow[rr] \arrow[rrrr, bend right, "\mathrm{id}"'] & & x \arrow[rr, "r"'] & & y
\end{tikzcd}
$$

from which we remove w to obtain

$$
\begin{tikzcd}
\bullet \arrow[r, "\ell"] \arrow[rr, bend left, "\mathrm{id}"] \arrow[dd, "f"'] \arrow[dr] \arrow[ddr] & \bullet \arrow[r] \arrow[d, "u"] \arrow[ddr] & \bullet \arrow[dd, "f"] \\
& \bullet \arrow[d, "v"] \arrow[dr] & \\
\bullet \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & x \arrow[r, "r"'] & y
\end{tikzcd}
\tag{16}
$$

(which is again equivalent to the previous one, by an argument involving inner anodyne extensions). The commutative square with sides u and f admits a lift, giving rise to the diagram

$$
\begin{tikzcd}
\bullet \arrow[r, "\ell"] \arrow[rr, bend left, "\mathrm{id}"] \arrow[dd, "f"'] \arrow[dr] \arrow[ddr] & \bullet \arrow[r] \arrow[d, "u"] & \bullet \arrow[dd, "f"] \\
& \bullet \arrow[d, "v"] \arrow[dr] \arrow[ur, "s"'] & \\
\bullet \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & x \arrow[r, "r"'] & y
\end{tikzcd}
\tag{17}
$$

(Again thinking about inner anodyne inclusions, we see that we do not have to include the 3-simplex exhibiting the composition of u, s, f.) Mapping Λ^3_1 to

$$
\begin{tikzcd}
\bullet \arrow[r, "\ell"] \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr] & \bullet \arrow[r] \arrow[d, "u"] & \bullet \\
& \bullet \arrow[ur, "s"'] &
\end{tikzcd}
$$

we may extend (17) by a 3-simplex via $\Lambda^3_1 \hookrightarrow \Delta^3$, from which we may remove $\Delta^{\{1\}}$ to obtain the diagram

$$
\begin{tikzcd}
\bullet \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] \arrow[d, "f"'] \arrow[dr] & \bullet \arrow[r, "s"] \arrow[d, "v"] \arrow[dr] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & x \arrow[r, "r"'] & y
\end{tikzcd}
$$

exhibiting f as the retract of a trivial fibration, so that f is a trivial fibration, and thus a weak equivalence. (Observe that only in this last step did we throw away information that cannot be recovered using anodyne extensions.)

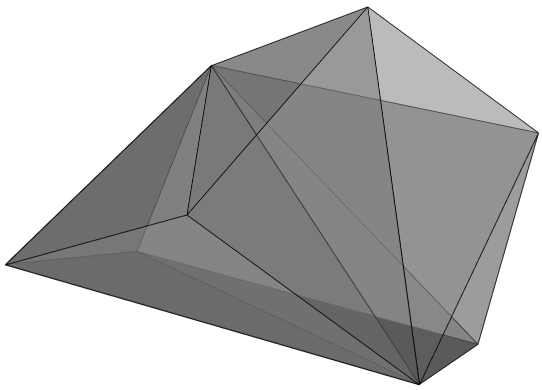
We now show that f in the diagram (15) is a weak equivalence without the assumption that it is a fibration. Consider the outer square of (15):

$$\begin{tikzcd}
\bullet \arrow[r, "\mathrm{id}"] \arrow[d, "f"'] \arrow[dr] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r, "\mathrm{id}"] & y
\end{tikzcd}$$

Factoring f into a trivial cofibration g followed by a fibration h, yields the diagram

$$\begin{tikzcd}
\bullet \arrow[r, "\mathrm{id}"] \arrow[d, "g"'] \arrow[dr] \arrow[dd, bend right, "f"'] & \bullet \arrow[d, "g"] \arrow[dd, bend left, "f"] \\
\bullet \arrow[r, "\mathrm{id}"] \arrow[d, "h"'] \arrow[dr] & z \arrow[d, "h"] \\
\bullet \arrow[r, "\mathrm{id}"] & y
\end{tikzcd}$$

Completing the diagram (15) and the above diagram to diagrams indexed by $\Delta^1 \times \Delta^2$ we can glue them along the face $\Delta^1 \times \Delta^{\{0,2\}}$ yielding a diagram with the following shape:



We wish to exhibit the objects x, y, z as apices of cocones on $\bullet \xleftarrow{g} \bullet \xrightarrow{\ell} \bullet$ together with cocone morphisms $x \to y \leftarrow z$. In the bottom left we have two 3-simplices exhibiting, respectively, the compositions of $\bullet \xrightarrow{f} \bullet \xrightarrow{\ell} x \xrightarrow{r} y$ and $\bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet \xrightarrow{\ell} x$, which are glued together along the faces $\Delta^{\{0,1,3\}}$ and $\Delta^{\{0,2,3\}}$, yielding an inner anodyne inclusion $\Delta^{\{0,2,3,4\}} \cup \Delta^{\{0,1,2,4\}} \subseteq \Delta^4$. Restricting along $\Delta^{\{0,1,3,4\}}$ produces a 3-simplex whitnessing the composition of $\bullet \xrightarrow{f} \bullet \xrightarrow{\ell g} x \xrightarrow{r} y$. Performing the same procedure yields a 3-simplex exhibiting the composition of $\bullet \xrightarrow{\ell} \bullet \xrightarrow{gr} z \xrightarrow{h} y$. These two 3-simplices together with the 3-simplices exhibiting the compositions of $\bullet \xrightarrow{\ell} \bullet \xrightarrow{w} x \xrightarrow{r} y$ and $\bullet \xrightarrow{g} \bullet \xrightarrow{\mathrm{id}} y \xrightarrow{h} z$ glued along their common faces produce the desired cocone morphisms $x \to y \leftarrow z$. Denote by c the (apex of the) colimit of

$\bullet \xleftarrow{g} \bullet \xrightarrow{\ell} \bullet$, then we obtain a square of cocones whose apices yield the lower right square in the diagram

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "g"'] \arrow[dr] \arrow[rr, bend left, "\mathrm{id}"] & \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r] \arrow[d, "h"'] \arrow[dr] & c \arrow[r] \arrow[d] \arrow[dr] & z \arrow[d, "h"] \\
\bullet \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & x \arrow[r] & y
\end{tikzcd}$$

together with a 2-simplex exhibiting the composition of $\bullet \to c \to z$ to id, which we have not indicated. The lower left and upper right squares are obtained from glueing 2-simplices coming from the diagram of cocones constructed as well as the two 4-simplices constructed above. The morphism g is a weak equivalence by assumption, as it is a trivial cofibration, and h is a weak equivalence, as it is a fibration, so that we may apply the argument from the beginning of the proof.

Conversely, assume that (C, F) satisfies (a) - (c), then we only need to show that $((C \cap W), F)$ and $(C, (F \cap W))$ form weak factorisation systems, which follows from the retract argument (Lemma B.0.7). $\square$

Proposition B.0.10. *Let (M, W) be a relative ∞-category with finite limits and colimits, in which W satisfies the 2-out-of-3 property, then a pair (C, F) of collections of morphisms in M forms a model structure iff*

(a) *the pair $((C \cap W), F)$ (resp. $(C, F \cap W)$) forms a weak factorisation sytem,*

(b) *any morphisms in M factors as a morphism in C (resp. ${}^{\boxslash}F$) followed by a morphism in $C^{\boxslash}$ (resp. F), and*

(c) *$C^{\boxslash} \subseteq F \cap W$ (resp. ${}^{\boxslash}F \subseteq C \cap W$).*

Proof. Consider a morphism $x \to y$ in $F \cap W$, then we must show that it lies in $C^{\boxslash}$. First, factor $x \to y$ as a morphism $x \to y'$ in C, followed by a morphism $y' \to y$ in $C^{\boxslash}$. By assumption $y' \to y$ is in W, so that by the 2-out-of-3 property $x \to y'$ is in W, and the lifting problem

$$\begin{tikzcd}
x \arrow[r] \arrow[d] & x \arrow[d] \\
y' \arrow[r] & y
\end{tikzcd}$$

admits a solution $y' \to x$, as $x \to y$ is in F. Then $y' \to x$ may be used to construct a retract

$$\begin{tikzcd}
x \arrow[r] \arrow[d] & y' \arrow[r] \arrow[d] & x \arrow[d] \\
y \arrow[r] & y \arrow[r] & y,
\end{tikzcd}$$

so that $x \to y$ is contained in $C^{\boxslash}$ by Proposition B.0.5. $\square$

C Pro-objects in ∞-categories

Here we collect some useful properties of pro-objects in ∞-categories used in §2.3.1.

Proposition C.0.1. *Let C be an ∞-category admitting finite products, then $\mathrm{Pro}(C)$ admits finite products.*

Proof. Let $x_0, \dots, x_n$ be objects in $\mathrm{Pro}(C)$, then for each $0 \leq i \leq n$ there exists a filtered small ordinary category A_i and a functor $x_{i\bullet} : A_i \to C$ such that $x_i \simeq$ "$\lim_{\alpha \in A_i}$"$x_{i\alpha}$ (see [Lur09, Prop. 5.3.1.16]). The category $A_0 \times \cdots \times A_n$ is filtered, and we claim that "$\lim_{\alpha \in A_0 \times \cdots \times A_n}$"$x_{0\bullet} \times \cdots \times x_{n\bullet}$ pro-represents the product of $x_0, \dots, x_n$. To see this, let y be any objects of C, then the isomorphisms

$$
\begin{aligned}
\mathrm{Pro}(C)(\text{“}\lim_{A_0 \times \cdots \times A_n}\text{”} x_{0\bullet} \times \cdots \times x_{n\bullet}, y) &\simeq \lim_{A_0 \times \cdots \times A_n} C(x_{0\bullet} \times \cdots \times x_{n\bullet}, y) \\
&\simeq \lim_{A_0 \times \cdots \times A_n} C(x_{0\bullet}, y) \times \cdots \times C(x_{n\bullet}, y) \\
&\simeq \lim_{A_0} \cdots \lim_{A_n} C(x_{0\bullet}, y) \times \cdots \times C(x_{n\bullet}, y) \\
&\simeq \lim_{A_0} \cdots \lim_{A_{n-1}} C(x_{0\bullet}, y) \times \cdots \times C(x_{n-1\bullet}, y) \times C(x_n, y) \\
&\cdots \\
&\simeq \lim_{A_0} C(x_{0\bullet}, y) \times C(x_1, y) \times \cdots \times C(x_n, y) \\
&\simeq C(x_0, y) \times \cdots \times C(x_n, y)
\end{aligned}
$$

are natural in y. $\square$

Lemma C.0.2. *Let I be a set, and for each element $i \in I$ consider a small filtered category A_i and a functor $X_i : A_i \to \mathcal{S}$, then the canonical morphism*

$$\operatorname*{colim}_{(\alpha_i) \in \prod A_i} \prod_{i \in I} X_{i,\alpha_i} \to \prod_{i \in I} \operatorname*{colim}_{\alpha_i \in A_i} X_{i,\alpha_i} \tag{18}$$

is an equivalence.

Proof. By [KS06, Prop. 3.1.11.ii] the statement is true in **Set**. Then, by [Cis19, Cor. 7.9.9] we may lift the functors $X_i : A_i \to \mathcal{S}$ to functors $A_i \to \widehat{\Delta}$, which we may then compose with the Ex^∞ functor to obtain functors valued in Kan complexes. The morphism in $\widehat{\Delta}$ corresponding to (18) is then an isomorphism, and the statement follows from the fact that Kan complexes as well as weak equivalences are closed under filtered colimits (see [Cis19, Lm. 3.1.24 & Cor. 4.1.17]). $\square$

Proposition C.0.3. *Let C be an accessible ∞-category admitting finite limits and coproducts, then the ∞-category $\mathrm{Pro}(C)$ is cocomplete.*

Proof. We show that $\mathrm{Pro}(C)$ admits pushouts and small coproducts.

$\underline{\mathrm{Pro}(C)\text{ admits pushouts:}}$ Recall that $\mathrm{Pro}(C)$ may be identified with the full subcategory of $[C, \mathcal{S}]^{\mathrm{op}}$ spanned by the left exact functors $f : C \to \mathcal{S}$ such that $C_{/f}$ is accessible by [DAG XIII, Prop. 3.1.6]. Consider a pullback square

$$
\begin{tikzcd}
p \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & f \arrow[d] \\
g \arrow[r] & h
\end{tikzcd}
$$

of functors in $[C, \mathcal{S}]$ with f, g, h in $\mathrm{Pro}(C)$. As limits of functors are computed pointwise, $p : C \to \mathcal{S}$ commutes with finite limits. Moreover, the above diagram induces a homotopy pullback diagram

$$\begin{tikzcd} C^{\mathrm{op}}_{/p} \arrow[r] \arrow[d] & C^{\mathrm{op}}_{/f} \arrow[d] \\ C^{\mathrm{op}}_{/g} \arrow[r] & C^{\mathrm{op}}_{/h} \end{tikzcd}$$

in $\widehat{\Delta}$ w.r.t. the Joyal model structure. The morphisms $C^{\mathrm{op}}_{/f} \to C^{\mathrm{op}}_{/g}$ and $C^{\mathrm{op}}_{/h} \to C^{\mathrm{op}}_{/g}$ are colimit preserving, so that $C_{/p}$ is accessible by [Lur09, Prop. 5.4.6.6].

Pro(C) admits small coproducts: Let I be a small set, and consider a family of objects $x_\bullet : I \to \mathrm{Pro}(C)$, then for each i there exists a filtered small ordinary category A_i and a functor $x_{i\bullet} : A_i \to C$ such that $x_i \simeq$ "$\lim\limits_{\alpha \in A_i}$"$x_{i\alpha}$ (see [Lur09, Prop. 5.3.1.16]). By Lemma C.0.2 we obtain the canonical isomorphisms

$$\coprod_{i \in I} x_i \simeq \coprod_{i \in I} \text{“}\lim_{\alpha_i \in A_i}\text{”} x_{i\alpha_i} \simeq \text{“}\lim_{(\alpha_i) \in \prod A_i}\text{”} \coprod_{i \in I} x_{i\alpha_i},$$

in $\mathrm{Pro}(C)$, as limits and colimits in presheaf categories are computed pointwise. $\square$

Conventions and notation

- The term ∞-*category* means *quasi-category*.
- We identify ordinary categories with their nerves, and consequently do not notationally distinguish between ordinary categories and their nerves.
- $[\_, \_]$ denotes the internal hom in $\widehat{\Delta}$, the category of simplicial sets.
- Let C, D be ∞-categories, and $W \subseteq C$, a subcategory, then $[C, D]_W$ denotes the subcategory of $[C, D]$ spanned by those functors sending every morphism in W to an isomorphism.
- Let X be a simplicial set, then $X_\simeq$ denotes the classifying space of X, given e.g. by $\mathrm{Ex}^\infty A$.
- ∞-categories (including ordinary categories) are denoted by C, D, ...
- Let C be an ∞-category and let $x, y \in C$ be two objects, then the homotopy type of morphisms from x to y is denoted by $C(x, y)$.
- A final object in an ∞-category C is denoted by $\mathbf{1}_C$, or simply by $\mathbf{1}$, when C is clear from context.
- For any Cartesian closed ∞-category C and any two objects x, y in C the internal hom object in C is denoted by $\underline{C}(x, y)$ or sometimes y^x.
- For any ∞-category C we denote its subcategory of n-truncated objects by $C_{\leq n}$.
- For any ∞-category C with finite products and any group object G in C, we denote C_G the category of G-objects in C.

- For A any small ordinary category $\widehat{A}$ denotes the category of (set-valued) presheaves on A.
- For any two categories C, D, an arrow $C \hookrightarrow D$ denotes a fully faithful functor.
- We use the following notation for various ∞-categories:
  - Δ denotes the category of simplices. Its objects are denoted by Δ^n or $[n]$, depending on context.
  - $\square$ denotes the category of cubes.
  - $\mathcal{S}$ denotes the ∞-categories of homotopy types.
  - **Cat** denotes the ∞-category of ∞-categories.
  - $\mathbf{Cat}_{(1,1)}$ denotes the $(2,1)$-category of ordinary categories.
  - $\mathbf{Cat}'_{(1,1)}$ denotes the relative ordinary category of ordinary categories, with weak equivalences given by equivalences of ordinary categories.
  - **Top** denotes the ∞-category of ∞-toposes.
  - **Set** denotes the category of sets.
  - **TSpc** denotes the category of topological spaces.
  - Δ**TSpc** is the full subcategory of **TSpc** spanned by the Δ-generated topological spaces.
  - $\mathbf{Mfd}^r$ denotes the category of r-times differentiable smooth manifolds and smooth maps.
  - $\mathbf{Cart}^r$ denotes the full subcategory of $\mathbf{Mfd}^r$ spanned by the spaces of $\mathbf{R}^n$ $(0 \leq n < \infty)$.
  - $\mathbf{Diff}^r$ denotes, equivalently, the ∞-category of sheaves on $\mathbf{Mfd}^r$ or $\mathbf{Cart}^r$.
- We denote ∞-toposes by $\mathcal{E}, \mathcal{F}, \ldots$, when they are thought of as ambient settings in which to do geometry, and by $\mathcal{X}, \mathcal{Y}, \ldots$, when they are thought of as geometric objects in their own right.
- Canonical isomorphisms are often denoted by equality signs. (An isomorphism is canonical if it originates from a universal property. More precisely, let $u : X \to C$ be a right fibration, and x, x' two final objects in X, then for any morphism $x \to x'$ the morphism $ux \to ux'$ is a canonical isomorphism, and we may write $x = x'$.)

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