Differentiable sheaves II: Local contractibility and cofinality

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Abstract

Many important theorems in differential topology relate properties of manifolds to properties of their underlying homotopy types – defined e.g. using the total singular complex or the Čech complex of a good open cover. Upon embedding the category of manifolds into the ∞-topos $\mathbf{Diff}^\infty$ of differentiable sheaves one gains a further notion of underlying homotopy type: the *shape* of the corresponding differentiable sheaf.

We develop a theory of nerves for locally contractible ∞-toposes as well as broadly applicable recognition principles for when these calculate shapes. Using this theory we show that the notions of underlying homotopy type of a differentiable sheaf alluded to above (as well as many others) indeed agree with the shape.

Finally, working with the ∞-topos $\mathbf{Diff}^0$ of sheaves on topological manifolds, we give new and conceptual proofs of some classical statements in algebraic topology. These include Dugger and Isaksen's hypercovering theorem, and Lurie's vast generalisation of the Seifert – Van Kampen theorem.

This is the second of three articles on the ∞-topos of sheaves on the category of manifolds.

Contents

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Introduction

Many important results about smooth manifolds such as the classification of compact surfaces or the Poincaré-Hopf theorem express differential topological properties in terms of suitably defined underlying homotopy types of smooth manifolds. Similarly, important invariants of smooth manifolds such as their de Rham cohomology only depend on their underlying homotopy type. If M is a smooth manifold, there are many ways to define its underlying homotopy type, e.g., one may take

1. its smooth total singular complex;
2. its underlying topological space;
3. or to a hypercover $\cdots \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \coprod \mathbf{R}^d \rightrightarrows \coprod \mathbf{R}^d \longrightarrow M$ (e.g., the Čech complex of a good open cover of M), one may associate the corresponding simplicial set obtaining by replacing every copy of $\mathbf{R}^d$ by $\mathbf{1}$, $\cdots \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \coprod \mathbf{1} \rightrightarrows \coprod \mathbf{1}$.

Unfortunately, these constructions suffer from at least two defects: 1. They all rely on specific models of homotopy types (i.e., simplicial sets or topological spaces). 2. None of these constructions are expressed in terms of a universal property.

These defects may be remedied by thinking of underlying homotopy types in terms of covering spaces: Let $\mathcal{E}$ be an ∞-topos and denote by $\pi : \mathcal{E} \to \mathcal{S}$ the unique geometric morphism to $\mathcal{S}$. Let X be an object in $\mathcal{E}$, then for any map $A \to B$ of homotopy types and any map $X \to \pi^*B$, the pullback square

$$\begin{array}{ccc} E & \longrightarrow & \pi^*A \\ \big\downarrow & \lrcorner & \big\downarrow \\ X & \longrightarrow & \pi^*B \end{array} \tag{1}$$

produces a covering space over X (see [Hoy18, Prop. 3.4]). Given a further pair consisting of a homotopy type B' and a morphism $X \to \pi^*B'$, as well as a map $B' \to B$ making diagram

$$\begin{array}{ccc} & & \pi^*B' \\ & \nearrow & \big\downarrow \\ X & & \\ & \searrow & \big\downarrow \\ & & \pi^*B \end{array}$$

commute, the covering space $E \to X$ may be obtained via the same construction from the morphism $A \times_B B' \to B'$. Thus, any covering space over X obtained from B as in (1) may be obtained from B' in the same way, and if there exists a universal morphism $X \to \pi^*C$ as above, then all covering spaces over X constructed as in (1) may be obtained from homotopy types over C. While $\mathcal{E}(X, \pi^*(\_))$ is not in general representable, it *is* pro-representable, so that $\mathcal{E} \leftarrow \mathcal{S} : \pi^*$ admits a formal left adjoint $\pi_! : \mathcal{E} \to \mathrm{Pro}(\mathcal{S})$; a colimit preserving functor which associates to any object X a pro-homotopy type called the *shape* of X. In many cases, large classes of covering spaces may be recovered from its shape (see [Hoy18, Thms. 3.13 & 4.3] and Remark 1.21). For example, when $\mathcal{E}$ is the ∞-topos of sheaves on the ∞-category of schemes w.r.t. the étale topology, then the shape coincides with the étale homotopy type, and the category of 0-truncated covering spaces over any scheme, can be recovered from its étale homotopy type (see [Hoy18, §5]).

For a simpler example, consider the ∞-topos $[A^{\mathrm{op}}, \mathcal{S}]$ of presheaves on a small ∞-category A. In this case, the shape functor forms a true left adjoint, i.e., it factors through $\mathcal{S} \hookrightarrow \mathrm{Pro}(\mathcal{S})$, and is given by $\mathrm{colim} : [A^{\mathrm{op}}, \mathcal{S}] \to \mathcal{S}$. A salient property of this example is that the shape of any representable object is contractible, and that $[A^{\mathrm{op}}, \mathcal{S}]$ is generated under colimits by objects of contractible shape. Moreover, for a second small ∞-category B together with a functor $u : A \to B$, we obtain a triple adjunction

$$[A^{\mathrm{op}}, \mathcal{S}] \; \begin{matrix} \xrightarrow{\quad u_! \quad} \\ \perp \\ \xleftarrow{\quad u^* \quad} \\ \perp \\ \xrightarrow{\quad u_* \quad} \end{matrix} \; [B^{\mathrm{op}}, \mathcal{S}] \tag{2}$$

where $u_! : [A^{\mathrm{op}}, \mathcal{S}] \to [B^{\mathrm{op}}, \mathcal{S}]$ always preserves shapes, and $[A^{\mathrm{op}}, \mathcal{S}] \leftarrow [B^{\mathrm{op}}, \mathcal{S}] : u^*$ preserves shapes precisely when $u : A \to B$ is initial (a.k.a. cofinal, a.k.a. coinitial, a.k.a. ...). In general, for any ∞-topos $\mathcal{E}$ the shape functor $\pi_! : \mathcal{E} \to \mathrm{Pro}(\mathcal{S})$ factors through $\mathcal{S} \hookrightarrow \mathrm{Pro}(\mathcal{S})$ iff $\mathcal{E}$ is generated under colimits by a (small) set of objects with contractible shape (see Proposition 1.17). Such ∞-toposes are called ***locally contractible***, and share many of the pleasant properties of presheaf ∞-categories.

We now return to the constructions described in points 1. - 3. above. Denote by $\mathbf{Diff}^r$ the ∞-topos of ***r-times differentiable sheaves*** — $\mathcal{S}$-valued sheaves w.r.t. the usual Grothendieck topology on the category of Cartesian spaces $\mathbf{R}^d$ ($d \geq 0$) and r-times differentiable maps between them. Observe that the category of r-times differentiable manifolds forms a full subcategory of $\mathbf{Diff}^r$. In the preceding article, [Clo24a], we equip $\mathbf{Diff}^r$ with the structure of a fractured ∞-topos. Here we use this structure to give a new proof that $\mathbf{Diff}^r$ is locally contractible: the shape of $\mathbf{R}^d$ coincides with the shape of its underlying topological space, which is seen to be contractible via a simple Galois theoretic proof (see Lemma 2.1).

We are able to make similar cofinality arguments for locally contractible ∞-toposes as for presheaf ∞-toposes (see §1.2.1). For example, (any number of variants of) the functor $u : \Delta \to \mathbf{Diff}^r$ sending $[n]$ to the standard simplex can be regarded as initial in an appropriate sense, and one obtains an adjunction

$$u_! : [\Delta^{\mathrm{op}}, \mathcal{S}] \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Diff}^r : u^* \tag{3}$$

in which both adjoints preserve shapes. Moreover, if $r \geq s$, the forgetful functor $\mathbf{Cart}^r \to \mathbf{Cart}^s$ induces a triple adjunction

$$\mathbf{Diff}^r \; \begin{matrix} \xrightarrow{\quad u_! \quad} \\ \perp \\ \xleftarrow{\quad u^* \quad} \\ \perp \\ \xrightarrow{\quad u_* \quad} \end{matrix} \; \mathbf{Diff}^s \tag{4}$$

analogous to (2), where again $u_!$ and u^* preserve shapes. If $s = 0$, then $u_!$ sends any manifold to its underlying topological space.

Finally, taking a hypercover $U_\bullet$ of M such that $U_n = \coprod \mathbf{R}^d$ for all $n \geq 0$, we observe that

$$\pi_! M \simeq \pi_! \operatorname*{colim}_{[n]\in\Delta}(U_n) \simeq \operatorname*{colim}_{[n]\in\Delta} \pi_!(U_n) \simeq \operatorname*{colim}_{[n]\in\Delta} \pi_! \left(\coprod \mathbf{R}^d\right) \simeq \operatorname*{colim}_{[n]\in\Delta} \coprod \pi_!(\mathbf{R}^d) \simeq \operatorname*{colim}_{[n]\in\Delta} \coprod \pi_! \mathbf{1}_{\mathcal{S}} \tag{5}$$

by descent and the fact that $\pi_! : \mathbf{Diff}^r \to \mathcal{S}$ preserves colimits, showing that the simplicial set associated to $U_\bullet$ indeed calculates the correct homotopy type. Applying (3) to point 1. above, (4) to 2., and (5) to 3., we obtain the following theorem (see §2.2):

Theorem A. *The homotopy types described in points 1. - 3. above are all equivalent to the shape of M.* $\square$

Observe that Theorem A is obtained without the use of smooth approximations of continuous functions.

We now explain how Theorem A may be applied to the homotopy theory of topological spaces. We set $r = 0$, so that $\mathbf{Diff}^0$ is the ∞-topos of sheaves on topological manifolds. Then, the topological realisation - total singular complex adjunction

$$|\_| : \widehat{\Delta} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{TSpc} : s \tag{6}$$

factors as

$$\widehat{\Delta} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{Diff}^0_{\leq 0} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{TSpc},$$

where the first map is obtained from the cosimplicial diagram consisting of the standard topological simplices, and the second adjunction is obtained from the inclusion $v : \mathbf{Cart}^0 \hookrightarrow \mathbf{TSpc}$. Thus, by Theorem A the singular homotopy type of any topological space X is given by the shape of v^*X. This observation allows us to give simple proofs of well-known theorems relating the descent and homotopy theory for topological spaces:

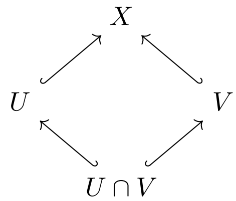
1. Lurie's Seifert – Van Kampen theorem (see [Lur17, Th. A.3.2], Theorem 2.29).

2. Dugger and Isaksen's hypercovering theorem (see [DI04, Th. 1.1], Theorem 2.35).

3. the fact that for any principal G-bundle $P \to B$, the topological space B is a homotopy quotient by the action of G on P (see Theorem 2.42).

To our knowledge, all previous proofs that $\mathbf{Diff}^r$ is locally contractible rely on modelling the shape functor by $v_! : \mathbf{Diff}^r \to \mathbf{TSpc}$, and then applying one of many variants of Dugger and Isaksen's hypercovering theorem, whose proofs are quite technical. Thus, in this article we turn this account on its head, by first providing a proof that $\mathbf{Diff}^r$ is locally contractible which eschews $v_! : \mathbf{Diff}^r \to \mathbf{TSpc}$, and then giving a proof of the hypercovering theorem which neatly relates singular shapes to descent. See Remark 2.5 for details and references.

To illustrate our techniques used to prove 1.-3. we consider a topological space X covered by open subsets U and V. A modern interpretation of the Seifert – Van Kampen theorem is that the square



(7)

induces a pushout square in $\mathcal{S}$. To prove this, all we need to do is to observe that (7) is carried to a pushout square in $\mathbf{Diff}^0$, and then the theorem follows from the fact that the shape functor $\pi_! : \mathbf{Diff}^0 \to \mathcal{S}$ preserves colimits.

Organisation

This article is a sequel to [Clo24a] in which we endow $\mathbf{Diff}^r$ with the structure of a fractured ∞-topos and discuss some consequences. Like [Clo24a], the present article has two parts, §1 & §2, where the first part refines some existing theory, in this case the theory of locally contractible ∞-toposes, and applies it to the ∞-topos $\mathbf{Diff}^r$.

1 Shapes and cofinality: As explained in the introduction, the *shape* of an object in an ∞-topos gives a Galoisic notion of its underlying pro-homotopy type. In §1.1 we first review shapes of ∞*-toposes* (rather than their objects); only afterwards, while discussing the functoriality of shapes, will we arrive at the notion of shapes of objects in an ∞-topos, and reconcile the two notions by noting that the shape of the final object of an ∞-topos is the same as the shape of the ∞-topos itself. Moreover, we give a cohomological criterion for when the shape of an ∞-topos is contractible. In §1.2 we specialise to locally contractible ∞-toposes — those ∞-toposes which are generated under colimits by a set of objects of contractible shape (so that the shape of any object is a homotopy type) — and in §1.2.1 we provide new recognition principles for when nerve diagrams (such as $\Delta \to \mathbf{Diff}^r$, giving rise to (3)) may be used to calculate these homotopy types. Finally, in §1.3 we show that the shape of the petit ∞-topos of any object in a fractured ∞-topos is equivalent to the shape of its gros ∞-topos, giving rise to a technique for exhibiting a fractured ∞-topos as locally contractible.

2 Shapes of differentiable sheaves: In §2.1 we use the techniques from §1.3 and [Clo24a, §2] to show that $\mathbf{Diff}^r$ is locally contractible: $\mathbf{Diff}^r$ is generated by the Cartesian spaces $\mathbf{R}^d$, whose shape may be calculated using its associated petit ∞-topos (which is just the ∞-topos of sheaves on its underlying topological space or $\mathbf{R}^d$), and may be shown to be contractible by combining the cohomological criterion from §1.1 with the fact that covering spaces on $\mathbf{R}^d$ are trivial. In §2.2 we use the techniques of §1.2 to prove Theorem A. Then in §2.3 we give two more applications of the technology developed so far: in §2.3.1 we show how Carchedi's calculation of the shape of the Haefliger stack seems almost inevitable using the calculus of shapes on $\mathbf{Diff}^r$ developed here, and in §2.3.2 we give elementary new proofs of several Seifert – Van Kampen like theorems, such as Dugger and Isaksen's hypercovering theorem.

In the final article in this series, [Clo24b], we further develop homotopical calculi on locally contractible ∞-toposes, and construct such calculi on $\mathbf{Diff}^r$ in order to study shapes of mapping sheaves.

Relation to other work

The main player in this article is arguably the shape functor $\pi_! : \mathbf{Diff}^r \to \mathcal{S}$. A sketch of the existence of this functor was first provided by Dugger in [Dug01, Prop. 8.3], and a complete construction was first given in [Sch13, Prop. 4.4.6]. Other constructions are given in [Car16, §3], [BEBP19, Prop. 1.3], [Bun22], [ADH21, §4.3], [Pav22]. On a conceptual level we drew inspiration from [Shu18] which discusses the relationship between $\mathbf{Diff}^r$ and $\mathcal{S}$ in a type theoretic setting.

The theory we develop in §2 is heavily influenced by [Cis03] which in turn draws heavily on Pursuing Stacks ([Gro83]) and the further development of Grothendieck's ideas in [Mal05].

Moreover, we give new and simpler proofs of classical results relating the homotopy theory of topological spaces with descent such as [DI04, Th. 1.3] and [Lur17, Th. A.3.2] as well as some classical facts about (unstable) Borel equivariant homotopy theory.

1 Shapes and cofinality

As explained in the introduction, the shape of any object in an ∞-topos provides a Galois theoretic notion of its underlying pro-homotopy type. In §1.1 we give a definition of the shape of an ∞-topos, and give a cohomological criterion for when a geometric morphism induces an equivalence of shapes. Then, we discuss local shape equivalences — geometric morphisms satisfying an analogous property to initial functors. In §1.2 we specialise to locally contractible ∞-toposes — those ∞-toposes for which the shape of every object is a homotopy type. We then show how certain nerve diagrams in locally contractible ∞-toposes satisfying a cofinality condition may be used to calculate shapes. Finally, in §1.3 we discuss how the structure of a fractured ∞-topos interacts with the property of being locally contractible.

Throughout this section $\mathcal{E}, \mathcal{F}, \mathcal{X}, \mathcal{Y}$ denote ∞-toposes.

1.1 Basic definitions and properties

We begin by giving a definition of the shape of an ∞-topos, before moving on to a discussion of the functoriality of the shape construction.

Denote by $\pi : \mathcal{X} \to \mathcal{S}$ the unique geometric morphism with target $\mathcal{S}$. By [DAG XIII, Prop. 3.1.6] and [Lur09, Prop. 5.4.7.7] the copresheaf $(\pi_{\mathcal{X}})_* \circ \pi_{\mathcal{X}}^*$ may be identified with an object in $\mathrm{Pro}(\mathcal{S})$, called the ***shape*** of $\mathcal{X}$, and is denoted by $\Pi_\infty(\mathcal{X})$. The ∞-topos is said to have ***trivial shape*** if $\Pi_\infty(\mathcal{X}) = \mathbf{1}$. Observe that $\mathcal{X}$ has trivial shape iff $\mathcal{X} \leftarrow \mathcal{S} : \pi^*$ is fully faithful.

Any geometric morphism $f : \mathcal{X} \to \mathcal{Y}$ gives to a morphism of shapes $\Pi_\infty(\mathcal{X}) \to \Pi_\infty(\mathcal{Y})$ by composing

$$(\pi_{\mathcal{X}})_* \circ \pi_{\mathcal{X}}^* = \mathcal{X}(\mathbf{1}_{\mathcal{X}}, \pi_{\mathcal{X}}^*(\_)) = \mathcal{X}(f^*\mathbf{1}_{\mathcal{Y}}, \pi_{\mathcal{Y}}^* \circ f^*(\_)) \leftarrow \mathcal{Y}(\mathbf{1}_{\mathcal{Y}}, \pi_{\mathcal{Y}}^*(\_)) = (\pi_{\mathcal{Y}})_* \circ \pi_{\mathcal{Y}}^*. \tag{8}$$

It turns out to be surprisingly difficult to coherently extend Π_∞ to a functor $\mathbf{Top} \to \mathrm{Pro}(\mathcal{S})$. As $\mathbf{Top}$ admits all filtered limits (see [Lur09, Th. 6.3.3.1]), the functor $\mathbf{Top} \leftarrow \mathcal{S}$, $\mathcal{S}_{/A} \leftarrow\!\shortmid A$ extends to a functor $\mathbf{Top} \leftarrow \mathrm{Pro}(\mathcal{S})$. In the upcoming [Mar] the shape Π_∞ will be exhibited as the left adjoint of $\mathbf{Top} \leftarrow \mathrm{Pro}(\mathcal{S})$, thus not only showing that Π_∞ can be made functorial, but also exhibiting a universal property of Π_∞, and moreover providing a version of the Seifert – Van Kampen theorem, as Π_∞ preserves colimits.

Fortunately, we will only require "local functoriality" of Π_∞: The functor $\mathcal{E} \leftarrow \mathcal{S} : \pi^*$ extends to a functor $\mathcal{E} \leftarrow \mathrm{Pro}(\mathcal{S})$. Tracing through the proof of [Cis19, Prop. 6.3.9] and again applying [DAG XIII, Prop. 3.1.6] and [Lur09, Prop. 5.4.7.7] one sees that this functor admits a left adjoint given by $X \mapsto \mathcal{E}(X, \pi^*(\_))$, which we denote by $(\pi_{\mathcal{E}})_! : \mathcal{E} \to \mathrm{Pro}(\mathcal{S})$ (or $\pi_!$, when $\mathcal{E}$ is clear from context). The two shape functors are compatible in that we recover $(\pi_{\mathcal{E}})_!$ from Π_∞ as the composition of $\mathcal{E} \xrightarrow{E \mapsto \mathcal{E}_{/E}} \mathbf{Top} \xrightarrow{\Pi_\infty} \mathrm{Pro}(\mathcal{S})$. Thus,

$(\pi_{\mathcal{E}})_!$ can also be viewed as satisfying a version of the Seifert – Van Kampen theorem, either by observing that it is a left adjoint or by considering its composition with Π_∞.

The shape of an ∞-topos is a powerful invariant, motivating the following definition:

Definition 1.1. A geometric morphism $f : \mathcal{X} \to \mathcal{Y}$ is called a ***shape equivalence*** if $\Pi_\infty f$ is an isomorphism. ⌟

Example 1.2. A functor $A \to B$ between small ∞-categories induces an equivalence between homotopy types $A_\simeq \xrightarrow{\cong} B_\simeq$ iff the induced geometric morphism $[A^{\mathrm{op}}, \mathcal{S}] \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} [B^{\mathrm{op}}, \mathcal{S}]$ is a shape equivalence. ⌟

We have the following cohomological Whitehead theorem for *hypercomplete* ∞-toposes:

Proposition 1.3. *If $\mathcal{X}, \mathcal{Y}$ are hypercomplete, then a geometric morphism $f : \mathcal{X} \to \mathcal{Y}$ is a shape equivalence iff the induced morphism*

$$H^i(\mathcal{X}; E) \leftarrow H^i(\mathcal{Y}; E)$$

is an isomorphism for all $i \geq 0$ and all E, where E is a set for $i = 0$, a group for $i = 1$, and an Abelian group for $i \geq 2$.

Proof. The only-if-statement is obvious. For the if-statement we want to prove that for any homotopy type K the induced map $\mathcal{X}\big(\mathbf{1}_\mathcal{X}, (\pi_\mathcal{X})^*(K)\big) \leftarrow \mathcal{Y}\big(\mathbf{1}_\mathcal{Y}, (\pi_\mathcal{Y})^*(K)\big)$ is an equivalence. First, we observe that it is enough to show this for the special case when K is n-truncated for some $n \in \mathbf{N}$, because for general K we then have

$$\begin{aligned}
\mathcal{X}\big(\mathbf{1}_\mathcal{X}, (\pi_\mathcal{X})^*(K)\big) &= \mathcal{X}\big(\mathbf{1}_\mathcal{X}, \lim_i\, (\pi_\mathcal{X})^*(K)_{\leq i}\big) \\
&= \mathcal{X}\big(\mathbf{1}_\mathcal{X}, \lim_i\, (\pi_\mathcal{X})^*(K_{\leq i})\big) \\
&= \lim_i \mathcal{X}\big(\mathbf{1}_\mathcal{X}, (\pi_\mathcal{X})^*(K_{\leq i})\big) \\
&= \lim_i \mathcal{Y}\big(\mathbf{1}_\mathcal{Y}, (\pi_\mathcal{Y})^*(K_{\leq i})\big) \\
&= \mathcal{Y}\big(\mathbf{1}_\mathcal{Y}, \lim_i\, (\pi_\mathcal{Y})^*(K_{\leq i})\big) \\
&= \mathcal{Y}\big(\mathbf{1}_\mathcal{Y}, \lim_i\, (\pi_\mathcal{Y})^*(K)_{\leq i}\big) \\
&= \mathcal{Y}\big(\mathbf{1}_\mathcal{Y}, (\pi_\mathcal{Y})^*(K)\big),
\end{aligned}$$

where the first and last isomorphisms follow from the hypercompleteness assumption, and the second and penultimate isomorphisms follow from [Lur09, 5.5.6.28].

We prove the statement for i-truncated K via induction on i: The base case $i = 0$ holds by assumption. Let $i > 0$, and assume the statement holds for all k-truncated objects, for $0 \leq k < i$. Let K be an i-truncated homotopy type, then we obtain the commutative square

$$\begin{array}{ccc}
\mathcal{X}\big(1_\mathcal{X}, (\pi_\mathcal{X})^*(K)\big) & \longleftarrow & \mathcal{Y}\big(1_\mathcal{Y}, (\pi_\mathcal{Y})^*(K)\big) \\
\downarrow & & \downarrow \\
\mathcal{X}\big(1_\mathcal{X}, (\pi_\mathcal{X})^*(K_{\leq i-1})\big) & \longleftarrow & \mathcal{Y}\big(1_\mathcal{Y}, (\pi_\mathcal{Y})^*(K_{\leq i-1})\big)
\end{array}$$

in which the bottom arrow is an isomorphism by the induction hypothesis. To show that the top horizontal morphism is an equivalence it is thus enough to show that for every fibre L of $K \to K_{\leq i-1}$ the map

$\mathcal{X}\big(\mathbf{1}_\mathcal{X}, (\pi_\mathcal{X})^*(L)\big) \leftarrow \mathcal{Y}\big(\mathbf{1}_\mathcal{Y}, (\pi_\mathcal{Y})^*(L)\big)$ is an equivalence, as $1 = \mathcal{X}\big(\mathbf{1}_\mathcal{X}, (\pi_\mathcal{X})^*(1)\big) \xleftarrow{=} \mathcal{Y}\big(\mathbf{1}_\mathcal{Y}, (\pi_\mathcal{Y})^*(1)\big) = 1$, and both $\mathcal{X}\big(\mathbf{1}_\mathcal{X}, (\pi_\mathcal{X})^*(\_)\big)$ and $\mathcal{Y}\big(\mathbf{1}_\mathcal{Y}, (\pi_\mathcal{Y})^*(\_)\big)$ preserve finite limits. We check the equivalence on connected components and on loop spaces. For every point in L we have

$$\begin{aligned}\Omega\,\mathcal{X}\big(\mathbf{1}_\mathcal{X}, (\pi_\mathcal{X})^*(L)\big) &= \mathcal{X}\big(\mathbf{1}_\mathcal{X}, (\pi_\mathcal{X})^*(\Omega L)\big) \\ &= \mathcal{Y}\big(\mathbf{1}_\mathcal{Y}, (\pi_\mathcal{Y})^*(\Omega L)\big) \\ &= \Omega\,\mathcal{Y}\big(\mathbf{1}_\mathcal{Y}, (\pi_\mathcal{X})^*(L)\big),\end{aligned}$$

where the second isomorphism follows from the induction hypothesis. On connected components we have

$$\begin{aligned}\pi_0\,\mathcal{X}\big(\mathbf{1}_\mathcal{X}, (\pi_\mathcal{X})^*(L)\big) &= H^i(\mathcal{X}, L) \\ &= H^i(\mathcal{Y}, L) \\ &= \pi_0\,\mathcal{Y}\big(\mathbf{1}_\mathcal{Y}, (\pi_\mathcal{Y})^*(L)\big)\end{aligned}$$

where the second isomorphism follows by assumption. □

Corollary 1.4. *Let $\mathcal{X}$ be a hypercomplete ∞-topos, then the shape of $\mathcal{X}$ is contractible iff the canonical map*

$$E \to H^0(\mathcal{X}, E)$$

is an equivalence for all sets E, and

$$H^i(\mathcal{X}, G) = 0$$

for all i and all G, where G is a group for $i = 1$, and an Abelian group for all $i \geq 2$. □

We now discuss how geometric morphisms satisfying extra conditions interact with shapes:

Definition 1.5. A geometric morphism $u : \mathcal{E} \to \mathcal{F}$ is called ***essential*** if u^* admits an extra left adjoint, which we denote by $u_!$. ⌟

Example 1.6. Any étale geometric morphism is essential. ⌟

Proposition 1.7. *Let $u : \mathcal{E} \to \mathcal{F}$ be an essential geometric morphism, then $u_!$ preserves shapes.*

Proof. The functors $(\pi_\mathcal{F})_! \circ u_!$ and $(\pi_\mathcal{E})_!$ are both left adjoint to the extension of the functor $\pi_\mathcal{E}^*$ to $\mathrm{Pro}(\mathcal{S}) \to \mathcal{E}$. □

Example 1.8. Let $u : A \to B$ be a functor between small ∞-categories, then $u_! : [A^{\mathrm{op}}, \mathcal{S}] \to [B^{\mathrm{op}}, \mathcal{S}]$ preserves shapes. ⌟

We now turn to a notion of cofinality in the toposic context. Let $f : \mathcal{E} \to \mathcal{F}$ be a geometric morphism, then by [Cis19, 6.4.2] the functor (8) may be extended to a base change map

$$\begin{array}{ccc} \mathcal{E} & \xleftarrow{\quad f^* \quad} & \mathcal{F} \\ & \Longrightarrow & \\ {\scriptstyle (\pi_\mathcal{E})_!} \searrow & & \swarrow {\scriptstyle (\pi_\mathcal{F})_!} \\ & \mathrm{Pro}(\mathcal{S}) & \end{array} \tag{9}$$

given by

$$(\pi_{\mathcal{E}})_! f^*Y = \mathcal{E}(f^*Y, (\pi_{\mathcal{E}})^*(\_)) = \mathcal{E}(f^*Y, f^* \circ (\pi_{\mathcal{F}})^*(\_)) \leftarrow \mathcal{F}(Y, (\pi_{\mathcal{F}})^*(\_)) = (\pi_{\mathcal{F}})_! Y$$

or equivalently by

$$\Pi_\infty(f_{/Y}) : \Pi_\infty(\mathcal{E}_{/f^*Y}) \to \Pi_\infty(\mathcal{F}_{/Y}).$$

Definition 1.9. The geometric morphism $f : \mathcal{E} \to \mathcal{F}$ is a ***local shape equivalence*** iff the base change map $(\pi_{\mathcal{E}})_! \circ f^* \Rightarrow (\pi_{\mathcal{F}})_!$ in (9) is an equivalence. ⌟

Example 1.10. A functor $A \to B$ between small ∞-categories is initial iff the induced geometric morphism $[A^{\mathrm{op}}, \mathcal{S}] \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} [B^{\mathrm{op}}, \mathcal{S}]$ is a local shape equivalence. ⌟

We conclude this subsection with some useful properties of local shape equivalences:

Proposition 1.11. *Let $f : \mathcal{E} \to \mathcal{F}$ be a geometric morphism. Assume that $\mathcal{F}$ is generated under small colimits by a subcategory C, and that the base change map $(\pi_{\mathcal{E}})_!(f^*F) \leftarrow (\pi_{\mathcal{F}})_! F$ is an isomorphism for every object F in C, then f is a local shape equivalence.* □

Proposition 1.12. *Let $a : \mathcal{E} \hookrightarrow \mathcal{F}$ be a geometric embedding which is also a local shape equivalence, then $(\pi_{\mathcal{E}})_! = (\pi_{\mathcal{F}})_! \circ a_*$.*

Proof. By assumption $(\pi_{\mathcal{E}})_! \circ a^* = (\pi_{\mathcal{F}})_!$, so the corollary follows from precomposing with a_*. □

Proposition 1.13. *Any geometric morphism $f : \mathcal{E} \to \mathcal{F}$ such that f^* is fully faithful is a local shape equivalence.*

Proof. For every Y in $\mathcal{F}$ we have $\mathcal{E}(f^*Y, \pi_{\mathcal{E}}^*(\_)) = \mathcal{E}(f^*Y, f^*\pi_{\mathcal{F}}^*(\_)) = \mathcal{F}(Y, \pi_{\mathcal{F}}^*(\_))$. □

Corollary 1.14. *If $\mathcal{E}$ has trivial shape, then $\pi_{\mathcal{E}} : \mathcal{E} \to \mathcal{S}$ is a local shape equivalence.* □

Recall that $\mathcal{E}$ is ***local*** if the global sections functor $\pi_* : \mathcal{F} \to \mathcal{S}$ admits a right adjoint, which we denote by $\pi^!$.

Proposition 1.15. *Any local ∞-topos has trivial shape.*

Proof. The adjunction $\pi^! \vdash \pi_*$ is a geometric morphism, so that $\pi^!\pi_*$ is the direct image component of a geometric morphism $\mathcal{S} \to \mathcal{S}$ and thus equivalent to the identity. By [JM89, Lm. 1.3] the counit of the induced adjunction $\mathrm{Ho}(\pi_*) : \mathrm{Ho}(\mathcal{X}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Ho}(\mathcal{S}) : \mathrm{Ho}(\pi^!)$ is an isomorphism, therefore the counit of $\pi^! \vdash \pi_*$ is an isomorphism, and therefore, finally, the unit of $\pi_* \vdash \pi^*$ is an isomorphism. □

1.2 Locally contractible toposes

We now specialise to a class of ∞-toposes, for which the theory of shapes is particularly nice.

Definition 1.16. An object in $\mathcal{E}$ is called ***contractible*** if its shape is trivial. ⌟

Proposition 1.17 ([MW23, Prop. 5.2.3])**.** *The following are equivalent:*

(I) *The shape functor $\pi_! : \mathcal{E} \to \mathrm{Pro}(\mathcal{S})$ factors through $\mathcal{S}$.*

(II) *The ∞-topos $\mathcal{E}$ is generated under colimits by its subcategory of contractible objects.*

Proof. The implication (II) $\implies$ (I) follows from the fact that the inclusion $\mathcal{S} \hookrightarrow \mathrm{Pro}(\mathcal{S})$ commutes with colimits. To show (I) $\implies$ (II), let E be an object of $\mathcal{E}$, then $\pi_! E$ is the colimit of the constant diagram $\mathbf{1}$ indexed by $\pi_! E$. Thus, $\pi^* \pi_! E$ is the colimit of the constant diagram $\mathbf{1}$ indexed by $\pi_! E$ in $\mathcal{E}$, so that E is the colimit of the diagram $E \times_{\pi^*\pi_! E} \mathbf{1}$ indexed by $\pi_! E$, and $E \times_{\pi^*\pi_! E} 1$ has contractible shape by [Lur17, Prop. A.1.9]. ☐

Definition 1.18. An ∞-topos is called ***locally contractible*** if it satisfies the equivalent conditions of Proposition 1.17. ⌟

Remark 1.19. Locally contractible ∞-toposes are called *locally of constant shape* in [Lur17], and *locally ∞-connected* in [Hoy18]. ⌟

Proposition 1.20 ([Lur17, Prop. A.1.11])**.** *Assume that $\mathcal{E}$ is locally contractible, then the functor $(\pi_!)_{/\mathbf{1}_\mathcal{E}} : \mathcal{E}_{/\mathbf{1}_\mathcal{E}} = \mathcal{E} \to \mathcal{S}_{/\pi_!\mathbf{1}_\mathcal{E}}$ admits a fully faithful right adjoint.* ☐

Remark 1.21. In Proposition 1.20 the image of the left adjoint of $(\pi_!)_{/\mathbf{1}_\mathcal{E}} : \mathcal{E}_{/\mathbf{1}_\mathcal{E}} = \mathcal{E} \to \mathcal{S}_{/\pi_!\mathbf{1}_\mathcal{E}}$ is given by the subcategory of $\mathcal{E}$ spanned by the covering spaces of $\mathbf{1}_\mathcal{E}$.

By [Cis19, Prop. 7.11.2] we obtain the following corollary (with notation as in Proposition 1.20):

Corollary 1.22. *The functor $(\pi_!)_{/\mathbf{1}_\mathcal{E}} : \mathcal{E}_{/\mathbf{1}_\mathcal{E}} = \mathcal{E} \to \mathcal{S}_{/\pi_!\mathbf{1}_\mathcal{E}}$ exhibits $\mathcal{S}_{/\pi_!\mathbf{1}_\mathcal{E}}$ as the localisation of $\mathcal{E}$ along its shape equivalences.* ☐

Example 1.23. Let A be a small ∞-category, then the constant presheaf functor $[A^{\mathrm{op}}, \mathcal{S}] \leftarrow \mathcal{S}$ admits both a left and a right adjoint given by the colimit and limit functors respectively, so that $[A^{\mathrm{op}}, \mathcal{S}]$ is a locally contractible ∞-topos. We have $\operatorname{colim} \mathbf{1} = A_\simeq$, and the image of the fully faithful right adjoint to the functor $\operatorname{colim}_{/\mathbf{1}} : [A^{\mathrm{op}}, \mathcal{S}] \to \mathcal{S}_{/A_\simeq}$ is spanned by those presheaves on A carrying all morphisms in A to isomorphisms. ⌟

Informally, this right adjoint functor is given by sending any map $X \to \pi_!\mathbf{1}$ in $\mathcal{S}$ to the map $\mathbf{1} \times_{\pi^*\pi_!\mathbf{1}} \pi^* X \to \mathbf{1}$. For locally contractible ∞-toposes, this makes precise the idea explained in the introduction that $\pi_!\mathbf{1}$ is characterised by universally controlling the theory of covering spaces on $\mathbf{1}$. For similar statements for non-locally contractible ∞-toposes, see [Hoy18].

Before moving on to nerves, we briefly discuss equivariant homotopy theory in locally contractible ∞-toposes. Assume that $\mathcal{E}$ is locally contractible, then for any group object G in $\mathcal{E}$ the ∞-category $\mathcal{E}_G$ is an ∞-topos as it is equivalent to $\mathcal{E}_{/BG}$. If moreover the shape functor $(\pi_\mathcal{E})_! : \mathcal{E} \to \mathcal{S}$ preserves finite products, then we see from the left square of

$$\begin{array}{ccccc}
\mathcal{E}_G & \xrightarrow{\simeq} & \mathcal{E}_{/BG} & \longrightarrow & \mathcal{E} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{S}_{\pi_! G} & \xrightarrow{\simeq} & \mathcal{S}_{/B\pi_! G} & \longrightarrow & \mathcal{S}
\end{array}$$

that $((\pi_{\mathcal{E}_G})_!)_{/\mathbf{1}_{\mathcal{E}_G}}$ is given by the functor $\mathcal{E}_G \to \mathcal{S}_{\pi_! G}$ taking any object X to $(\pi_\mathcal{E})_! X$ with its induced $(\pi_\mathcal{E})_! G$-action. By composing the two horizontal morphisms on the top we see that the quotient functor

$\_/G : \mathcal{E}_G \to \mathcal{E}$ is the extra left adjoint of an étale geometric morphism, so that Proposition 1.7 yields the following result:

Proposition 1.24. *For any object X in $\mathcal{E}_G$ the comparison morphism $\pi_! X/\pi_! G \to \pi_!(X/G)$ is an isomorphism in $\mathcal{S}$.* □

1.2.1 Nerves

We now discuss the main tool for calculating shapes in this article. Until the end of §1.2.1, $\mathcal{E}$ denotes a locally contractible ∞-topos, and A, a small ∞-category together with a functor $u : A \to \mathcal{E}$.

Proposition 1.25. *If the image of $u : A \to \mathcal{E}$ is spanned by contractible objects then the functors* $\operatorname{colim} : [A^{\mathrm{op}}, \mathcal{S}] \to \mathcal{S}$ *and* $(\pi_{\mathcal{E}})_! \circ u_!$ *are canonically equivalent.*

Proof. By assumption the composition of $A \xrightarrow{u} \mathcal{E} \xrightarrow{\pi_!} \mathcal{S}$ is equivalent to the constant functor $a \mapsto \mathbf{1}$, so the equivalence is obtained by extending by colimits. □

In the following two statements $C \subseteq \mathcal{E}$ denotes a small subcategory spanned by contractible objects and generating $\mathcal{E}$ under colimits.

Proposition 1.26. *The shape functor $(\pi_{\mathcal{E}})_!$ is canonically equivalent to* $\operatorname{colim}_{c \in C} \mathcal{E}(c, \_)$.

Proof. We observe that $(\pi_{[C^{\mathrm{op}},\mathcal{S}]})_! c = \operatorname{colim} C(\_, c) = \mathbf{1}$ for every object c in C, and apply first Proposition 1.11 and then Proposition 1.12 to the geometric morphism $\mathcal{E} \to [C^{\mathrm{op}}, \mathcal{S}]$. □

Theorem 1.27. *Assume that $u : A \to \mathcal{E}$ factors (uniquely) through $C \hookrightarrow \mathcal{E}$, and that the functor $u : A \to C$ is initial, then the natural transformation*

$$\operatorname{colim} \circ u^* \to (\pi_{\mathcal{E}})_!$$

is an equivalence.

Moreover, both $u_!$ and u^* preserve shape equivalences, and induce an adjoint equivalence as indicated in the following diagram:

$$\begin{array}{ccc} [A^{\mathrm{op}}, \mathcal{S}] & \underset{u^*}{\overset{u_!}{\rightleftarrows}}\ (\bot) & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{S}_{/A^{\simeq}} & \underset{\simeq}{\overset{\simeq}{\rightleftarrows}}\ (\bot) & \mathcal{S}_{/\pi_! \mathbf{1}} \end{array}$$

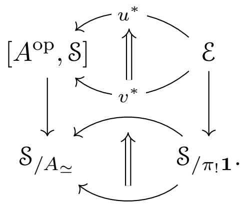
Proof. The two diagrams

$$\begin{array}{ccc} [A^{\mathrm{op}}, \mathcal{S}] & \xrightarrow{u_!} & \mathcal{E} \\ & {\scriptstyle \operatorname{colim}} \searrow \quad \swarrow {\scriptstyle \pi_!} & \\ & \mathcal{S} & \end{array} \qquad \begin{array}{ccc} [A^{\mathrm{op}}, \mathcal{S}] & \xleftarrow{u^*} & \mathcal{E} \\ & {\scriptstyle \operatorname{colim}} \searrow \quad \swarrow {\scriptstyle \pi_!} & \\ & \mathcal{S} & \end{array}$$

commute, the first one by Proposition 1.25, and the second one by the following calculation (obtained using Proposition 1.26): $(\pi_{\mathcal{E}})_! X = \operatorname{colim}_{c \in C} \mathcal{E}(c, X) = \operatorname{colim}_{a \in A} \mathcal{E}(ua, X)$. Thus, both $u_!$ and u^* preserve weak equivalences, inducing the indicated adjoint equivalence by Corollary 1.22 and [Cis19, Prop. 7.1.14]. □

We will often refer to any functor $u : A \to \mathcal{E}$ to which we intend to apply Theorem 1.27 as a ***nerve diagram***, and the functor $[A^{\mathrm{op}}, \mathcal{S}] \leftarrow \mathcal{E} : u^*$ as a ***nerve***. In the examples considered in this article, the functor $\alpha : A \to C$ usually induces a bijection on objects.

Remark 1.28. Let $u, v : A \to \mathcal{E}$ be two nerve diagrams satisfying the conditions of Theorem 1.27 (the small subcategories C are not assumed to be the same for u and v). Any natural transformation $u \to v$ induces a natural transformation $u^* \leftarrow v^*$, and by the universal property of localisations we obtain a diagram

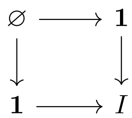


(10)

As the two functors $S_{/A_{\simeq}} \leftarrow \mathcal{S}_{/\pi_!\mathbf{1}}$ are equivalences they restrict to equivalences $A_{\simeq} \leftarrow \pi_!\mathbf{1}$. (This follows e.g. from [AF20, Th. 2.39], or the fact that $\mathcal{S}_{/\_} : \mathcal{S} \to \mathbf{Top}$ is fully faithful.) The functor from morphisms $A_{\simeq} \leftarrow \pi_!\mathbf{1}$ to colimit preserving functors $[A_{\simeq}^{\mathrm{op}}, \mathcal{S}] \leftarrow [(\pi_!\mathbf{1})^{\mathrm{op}}, \mathcal{S}]$ is fully faithful, and thus the lower natural transformation in (10) must be a natural isomorphism. ⌟

We will repeatedly use Propositions 1.32 & 1.33 below to verify the conditions of the above proposition.

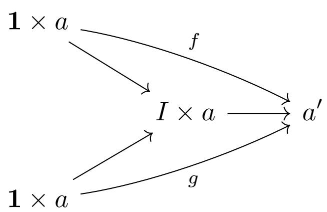
Definition 1.29 ([Mal05, §1.4])**.** Let A be an ordinary category admitting a final object $\mathbf{1}$, then an object I in A with two morphisms $\mathbf{1} \rightrightarrows I$ is called an ***interval*** in A. If A admits an initial object $\varnothing$, and the square



is a pullback, then I is ***separating interval***. ⌟

Example 1.30. Let $\mathcal{E}$ be an ordinary topos, then the subobject classifier Ω in $\mathcal{E}$ canonically has the structure of a separating interval. The first morphism $\mathbf{1} \to \Omega$ is given by the universal monomorphism, and the second morphism $\mathbf{1} \to \Omega$ classifies the subobject $\varnothing \to \mathbf{1}$. ⌟

Definition 1.31. Let A be an ordinary category equipped with an interval I, then an ***I-homotopy*** between two maps $f, g : a \rightrightarrows a'$ is a commutative diagram



and f and g are called ***I-homotopic*** if there exists an I-homotopy between f and g. A map $f : a \to a'$ is an ***I-homotopy equivalence*** if there exists a map $a \leftarrow a' : g$, such that gf and fg are I-homotopic to id_a and $\mathrm{id}_{a'}$, respectively. An object a in A is ***I-contractible***, if the unique morphism $a \to \mathbf{1}$ is an I-homotopy equivalence. ⌟

Proposition 1.32. *Let (A, I) and (B, J) be pairs consisting of small ordinary categories together with an interval, and let $u : A \to B$ be a functor carrying I to J (including the inclusions of the final object, which u must then preserve). Assume that*

(a) *$\pi_! : [A^{\mathrm{op}}, \mathcal{S}] \to \mathcal{S}$ preserves finite products, and that*

(b) *every object in B is J-contractible*

then u is initial.

Proof. The functor u is initial iff for every object b in B the shape of u^*b is contractible (see [Cis19, Cor. 4.4.31]). Let $J \times b \to b$ be an J-contraction of b, then the unit morphisms produce a diagram

$$\begin{array}{ccccc}
u^*b \cong \mathbf{1}_A \times u^*b & \longrightarrow & (u^*u_!\mathbf{1}_A \times u^*b) \cong u^*(\mathbf{1}_A \times b) & & \\
\downarrow & & \downarrow & \searrow^{\mathrm{id}} & \\
I \times u^*b & \longrightarrow & (u^*u_! I \times u^*b) \cong u^*(J \times b) & \longrightarrow & u^*b, \\
\uparrow & & \uparrow & \nearrow_{0} & \\
u^*b \cong \mathbf{1}_A \times u^*b & \longrightarrow & (u^*u_!\mathbf{1}_A \times u^*b) \cong u^*(\mathbf{1}_A \times b) & &
\end{array}$$

showing that u^*b is I-contractible in $[A^{\mathrm{op}}, \mathcal{S}]$. □

See [Clo24b, App. A] or [Cis06, §8.4] for some background on the cube category.

Proposition 1.33. *Let (B, J) be a pair consisting of a small ordinary category together with an interval. Let $u : \square \to B$ be a functor carrying the interval $\square^1$ to J (including the inclusions of the final object, which u must then preserve). If every object in B is J-contractible then u is initial.*

Proof. As in the previous proposition, the functor u is initial iff for every object b in B the shape of u^*b is contractible. We will require the following claim, which we prove below.
Claim: There exists a natural morphism of cubical sets $X_1 \otimes X_2 \to X_1 \times X_2$.

Let $J \times b \to b$ be an J-contraction of b, then the unit morphisms produce a diagram

$$\begin{array}{ccccccc}
u^*b \cong \mathbf{1}_\square \otimes u^*b & \longrightarrow & \mathbf{1}_\square \times u^*b & \longrightarrow & (u^*u_!\mathbf{1}_\square \times u^*b) \cong u^*(\mathbf{1}_\square \times b) & & \\
\downarrow & & \downarrow & & \downarrow & \searrow^{\mathrm{id}} & \\
\square^1 \otimes u^*b & \longrightarrow & \square^1 \times u^*b & \longrightarrow & (u^*u_!\square^1 \times u^*b) \cong u^*(J \times b) & \longrightarrow & u^*b, \\
\uparrow & & \uparrow & & \uparrow & \nearrow_{0} & \\
u^*b \cong \mathbf{1}_\square \otimes u^*b & \longrightarrow & \mathbf{1}_\square \times u^*b & \longrightarrow & (u^*u_!\mathbf{1}_\square \times u^*b) \cong u^*(\mathbf{1}_\square \times b) & &
\end{array}$$

showing that u^*b is $\square^1$-contractible because $\pi_!(X_1 \otimes X_2) \simeq \pi_! X_1 \times \pi_! X_2$ for all cubical sets X_1, X_2 (see [Cis06, Cor. 8.4.32]).

Proof of claim: For any two cubical sets X_1, X_2 there are canonical morphisms $X_1 \otimes X_2 \to X_i$ $(i = 1, 2)$. To see this, note that for any $k_1, k_2 \in \mathbf{N}$ we have projection maps (in **Set**) $\square^{k_1} \otimes \square^{k_2} \cong \{0,1\}^{k_1} \times \{0,1\}^{k_2} \to \{0,1\}^{k_i}$ for $i = 1, 2$; the canonical morphisms $X_1 \otimes X_2 \to X_i$ $(i = 1, 2)$ are then obtained by extending by colimits, yielding the desired morphism. □

Proposition 1.34. *With notation as in Proposition 1.33, if B admits products, $u : \square \to B$ is monoidal, and every object in B is a finite product of J, then u is initial if J is J-contractible.*

Proof. It is enough to show that if the objects b and b' are J contractible, then so is $b \times b'$. So let $J \times b \to b$ and $J \times b' \to b'$ be contractions, then the composition of $J \times b \times b' \to J \times J \times b \times b' \to b \times b'$ is a contraction of $b \times b'$, where the first morphism is induced by the diagonal morphism $J \to J \times J$. □

1.3 Fractured ∞-toposes and shapes

We now prove the result that will allow us to exhibit $\mathbf{Diff}^r$ as a locally contractible topos. This result may be viewed as a vast generalisation of the techniques underlying [Cis03, Lm. 6.1.5]. Throughout this subsection $j_! : \mathcal{E}^{\mathrm{corp}} \rightleftarrows \mathcal{E} : j^*$ denotes a fractured ∞-topos (see [Lur18, Ch. 20] and [Clo24a, §1]).

Theorem 1.35. *For any corporeal object X the geometric morphism $j_! : \mathcal{E}^{\mathrm{corp}}_{/X} \rightleftarrows \mathcal{E}_{/X} : j^*$ is a local geometric morphism.*

Proof. This is a consequence of property [Clo24a, (b)] of in the definition of fractured ∞-toposes, and Proposition 1.13. □

The following result could be viewed as a corollary of the above, but we note that it follows more immediately from Proposition 1.7.

Theorem 1.36. *The functor $j_! : \mathcal{E}^{\mathrm{corp}} \to \mathcal{E}$ preserves shapes.* □

Thus, the cohomology of a geometric object such as a scheme with coefficients in a locally constant sheaf is the same when computed in its gros or petit topos. For us, Theorem 1.35 provides a way of showing that a topos is locally contractible, as seen in the following corollary.

Corollary 1.37. *Let $C \subseteq \mathcal{E}^{\mathrm{corp}}$ be a small subcategory, spanned by contractible objects, and generating $\mathcal{E}^{\mathrm{corp}}$ under colimits, then $j_!C \subseteq \mathcal{E}$ is a small subcategory, spanned by contractible objects, generating $\mathcal{E}$ under colimits.* □

In other words, if $\mathcal{E}^{\mathrm{corp}}$ is locally contractible, then so is $\mathcal{E}$.

Remark 1.38. By [Lur18, Rmk. 20.3.2.6] the subcategory of $\mathcal{E}_{/X}$ spanned by admissible morphisms is an ∞-topos for *any* object X in $\mathcal{E}$, and it can be shown that the inclusion of this subcategory into $\mathcal{E}_{/X}$ induces an equivalence on shapes. ⌟

2 Shapes of differentiable sheaves

Fix an element r of $\mathbf{N} \cup \{\infty\}$ for the remainder of this article. Recall from [Clo24a, §2] that

1. $\mathbf{Mfd}^r$ denotes the category of r-times differentiable (2$^{\mathrm{nd}}$-countable, Hausdorff) manifolds and r-times differentiable maps, and
2. $\mathbf{Mfd}^r_{\text{ét}}$ denotes the category of r-differentiable manifolds and r-differentiable open embeddings.
3. $\mathbf{Cart}^r$ denotes the full subcategory of $\mathbf{Mfd}^r$ spanned by the spaces $\mathbf{R}^n$ $(0 \leq n < \infty)$.

4. $\mathbf{Cart}^r_{\text{ét}}$ denotes the full subcategory of $\mathbf{Mfd}^r_{\text{ét}}$ spanned by the spaces $\mathbf{R}^n$ $(0 \le n < \infty)$.

Each of these small categories is equipped with the Grothendieck topology in which a sieve on a manifold is a covering sieve iff it contains a covering consisting of jointly surjective open embeddings. An $\mathcal{S}$-valued sheaf on $\mathbf{Cart}^r$ (or equivalently $\mathbf{Mfd}^r$) is an ***r-times differentiable sheaf***, and the ∞-topos thereof is denoted by $\mathbf{Diff}^r$. Similarly, an $\mathcal{S}$-valued sheaf on $\mathbf{Cart}^r_{\text{ét}}$ (or equivalently $\mathbf{Mfd}^r_{\text{ét}}$) is an ***étale r-times differentiable stack***, and the ∞-topos thereof is denoted by $\mathbf{Diff}^r_{\text{ét}}$. Moreover, the left Kan extension $j_! : \mathbf{Diff}^r_{\text{ét}} \to \mathbf{Diff}^r$ of the canonical functor $j : \mathbf{Cart}^r_{\text{ét}} \to \mathbf{Cart}^r \hookrightarrow \mathbf{Diff}^r$ equips $\mathbf{Diff}^r$ with the structure of a *fractured ∞-topos* (see [Clo24a, §2.1]). For any r-times differentiable étale stack X its image under $j_!$ is usually likewise denoted by X.

We first prove in §2.1 that $\mathbf{Diff}^r$ is a locally contractible ∞-topos compatibly with its fractured ∞-topos structure, so that we may apply the technology of §1 to $\mathbf{Diff}^r$. In §2.2 we prove Theorem A from the introduction stating that various ways of extracting homotopy types from manifolds are equivalent. Finally, in §2.3 we discuss some applications of the technology developed so far; we give a streamlined account of Carchedi's calculation of the shape of the Haefliger stack in §2.3.1, and provide new, simpler proofs of classical descent theorems in algebraic topology such as Dugger and Isaksen's hypercovering theorem in §2.3.2.

2.1 $\mathbf{Diff}^r$ is a locally contractible ∞-topos

Lemma 2.1. *The shape of $\mathbf{R}^d$ is contractible in $\mathbf{Diff}^r_{\text{ét}}$ for every $d \in \mathbf{N}$.*

Proof. The ∞-topos $(\mathbf{Diff}^r_{\text{ét}})_{/\mathbf{R}^d}$ is equivalent to the ∞-topos of sheaves on the underlying topological space of $\mathbf{R}^d$. We will check that $(\mathbf{Diff}^r_{\text{ét}})_{/\mathbf{R}^d}$ is contractible (and moreover locally contractible) by induction on d.

The case $d = 0$ is clear.

Next, we check the case $d = 1$ using Corollary 1.4. Let X be a set, then $H^0(\mathbf{R}, X) = \mathbf{TSpc}(\mathbf{R}, X) = X$, as $\mathbf{R}$ is connected. Let G be any group, then $H^1(\mathbf{R}, G)$ is equivalent to the set of isomorphism classes of principle G-bundles on $\mathbf{R}$, which are constant (and thus all equivalent) by the standard argument that covering spaces on $\mathbf{R}$ are constant (see e.g., [Sch14, Lm. 5.1.3]). Finally, by [APG90, §II.6.2] $\mathbf{R}$ has covering dimension ≤ 1, and thus cohomological dimension ≤ 1 by the discussion following [Lur09, Rmk. 7.2.2.19]. (Alternatively, one may prove that $\mathbf{R}$ has cohomological dimension ≤ 1 using a similar argument to the one used to exhibit the triviality of covering spaces on $\mathbf{R}$, as shown in [Sch14, Lm. 5.1.1].)

Observe that $(\mathbf{Diff}^r_{\text{ét}})_{/\mathbf{R}}$ is moreover locally contractible, as $\mathbf{R}$ has a basis given by open intervals, which are each diffeomorphic to $\mathbf{R}$. Now, let $d > 1$, and assume that $(\mathbf{Diff}^r_{\text{ét}})_{/\mathbf{R}^{d-1}}$ has contractible shape and moreover is locally contractible, so that by Proposition 1.20 the adjunction $\pi_! : (\mathbf{Diff}^r_{\text{ét}})_{/\mathbf{R}^{d-1}} \rightleftarrows \mathcal{S} : \pi^*$ is a reflection. Now, observe that the triple adjunctions

$$(\mathbf{Diff}^r_{\text{ét}})_{/\mathbf{R}} \quad \begin{array}{c} \xrightarrow{\pi_!} \\ \xleftarrow{\pi^*} \\ \xrightarrow{\pi_*} \end{array} \quad \mathcal{S} \qquad (\mathbf{Diff}^r_{\text{ét}})_{/\mathbf{R}^{d-1}} \quad \begin{array}{c} \xrightarrow{\pi_!} \\ \xleftarrow{\pi^*} \\ \xrightarrow{\pi_*} \end{array} \quad \mathcal{S}$$

are reflective adjunctions in the $(\infty, 2)$-category of presentable ∞-categories and left adjoints, so that $(\mathbf{Diff}^r_{\text{ét}})_{/\mathbf{R}^d} \leftarrow \mathcal{S} : \pi^*$ is a reflective subcategory, as it is obtained from the tensor product of the above. □

Corollary 2.2. *The ∞-topos $\mathbf{Diff}^r_{\text{ét}}$ is locally contractible.* □

By Theorem 1.35 we then obtain the following corollary:

Corollary 2.3. *The shape of* $\mathbf{R}^d$ *is contractible in* $\mathbf{Diff}^r$ *for all* $d \in \mathbf{N}$. □

Corollary 2.4. *The* ∞*-topos* $\mathbf{Diff}^r$ *is locally contractible.* □

Remark 2.5. The functor $\pi_! : \mathbf{Diff}^r \to \mathcal{S}$ has been shown to exist many times before, e.g. in [Dug01, Prop. 8.3], [Sch13, §4.4], [Car16, Prop. 3.1], [BEBP19, Prop. 1.3], [Bun22], [ADH21, §4.3], [Pav22]. All of these sources rely on some variant of the nerve or Seifert – Van Kampen theorem (see [Bor48], [Ler50], [Wei52], [Seg68, §4], [DI04, Th. 1.1], [Lur17, Th. A.3.1]) to implement some version of the following argument: one shows that

1. colim : $[(\mathbf{Cart}^r)^{\mathrm{op}}, \mathcal{S}] \to \mathcal{S}$ sends covers to colimits, and
2. constant presheaves on $\mathbf{Cart}^r$ are sheaves,

so that the adjunction colim : $[(\mathbf{Cart}^r)^{\mathrm{op}}, \mathcal{S}] \rightleftarrows \mathcal{S}$: const restricts to $\pi_! : \mathbf{Diff}^r \rightleftarrows \mathcal{S} : \pi^*$. We will discuss the specific manifestation of this argument used in [Car16] in §2.3.1.

The proofs of many variants of the nerve and Seifert – Van Kampen theorem, in particular [DI04, Th. 1.1] and [Lur17, Th. A.3.1], are quite involved. We will obtain these practically for free in §2.3.2. ⌟

Corollary 2.6. *The shape functor* $\pi_! : \mathbf{Diff}^r \to \mathcal{S}$ *preserves finite products.*

Proof. By Proposition 1.12 the shape of any sheaf in $\mathbf{Diff}^r$ may be computed as the colimit of the corresponding presheaf on $\mathbf{Cart}^r$, but $\mathbf{Cart}^r$ has finite products, and is thus sifted. □

Several of the references listed in Remark 2.5 moreover show (some variant of) the following result:

Proposition 2.7. *The shape functor* $\pi_! : \mathbf{Diff}^r \to \mathcal{S}$ *exhibits* $\mathcal{S}$ *as the localisation of* $\mathbf{Diff}^r$ *along the projection map* $\mathbf{R}^1 \times X \to X$ *for all differentiable sheaves* X.

Proof. Denote by W the class of maps inverted by the localisation of $\mathbf{Diff}^r$ along the projection map $\mathbf{R}^1 \times X \to X$ for all differentiable sheaves X. By Corollary 2.6 these projection maps are all shape equivalences, so W is contained in the class of shape equivalence. On the other hand, the map $\mathbf{R}^d \to \mathbf{1}$ can be decomposed into a sequence of projection maps $\mathbf{R}^d \to \mathbf{R}^{d-1} \to \cdots \to \mathbf{R} \to \mathbf{1}$, and are thus in W, and therefore by the 2-out-of-3 property, all maps in $\mathbf{Cart}^r$ are in W. Any $\mathbf{R}$-local sheaf may be viewed as an $\mathbf{R}$-local presheaf on $\mathbf{Cart}^r$, and is thus in the image of $(\pi_{\mathbf{Cart}^r})^*$ by Example 1.23, and is thus in the image of $(\pi_{\mathbf{Diff}^r})^* = a^* \circ (\pi_{\mathbf{Cart}^r})^*$, where a^* is the sheafification functor. □

2.2 Comparing methods of calculating underlying homotopy types of differentiable sheaves

We first construct various nerve diagrams in $\mathbf{Diff}^r$ in §2.2.1, and show that the induced nerves all calculate shapes. Then, in §2.2.2 we show that sending any r-times differentiable manifold to its underlying s-times differentiable manifold for $r \geq s \geq 0$ does not change its shape.

2.2.1 Nerves

Here we consider five different nerve diagrams:

$\mathbf{A}^\bullet : \Delta \to \mathbf{Diff}^r_{\leq 0}$

$\Delta^\bullet_{\mathrm{sub}} : \Delta \to \mathbf{Diff}^r_{\leq 0}$

$\Delta^\bullet : \Delta \to \mathbf{Diff}^r_{\leq 0}$

$\boldsymbol{\square}^\bullet : \square \to \mathbf{Diff}^r_{\leq 0}$

$\square^\bullet : \square \to \mathbf{Diff}^r_{\leq 0}$

In each case we will use Theorem 1.27 to show that the five resulting nerves all calculate shapes.

Extended simplices

Definition 2.8. Consider the cosimplicial object

$$\begin{array}{rccl} \mathbf{A}^\bullet : & \Delta & \to & \mathbf{Diff}^r_{\leq 0} \\ & [n] & \mapsto & \mathbf{A}^n := \left\{ \ (x_0, \dots, x_n) \in \mathbf{R}^{n+1} \ \middle| \ x_0 + \dots + x_n = 1 \ \right\}, \end{array}$$

then the spaces $\mathbf{A}^n$ for $n \geq 0$ are referred to as ***extended simplices***. Moreover we write

$$\begin{array}{rcll} \partial \mathbf{A}^n & := & \mathbf{A}^\bullet_! \partial \Delta^n, & n \geq 0 \\ \mathbf{\Lambda}^n_k & := & \mathbf{A}^\bullet_! \Lambda^n_k, & n \geq 1, \ n \geq k \geq 0. \end{array}$$

⌟

Proposition 2.9. *The canonical natural transformation* $\operatorname{colim} \circ (\mathbf{A}^\bullet)^* \to (\pi_{\mathbf{Diff}^r})_!$ *is an equivalence.*

Proof. The image of $\mathbf{A}^\bullet$ is given by $\mathbf{Cart}^r$, and the induced functor $\Delta \to \mathbf{Cart}^r$ is easily seen to satisfy the conditions of Proposition 1.32, thus verifying the conditions of Theorem 1.27. □

Closed simplices Consider the cosimplicial object

$$\begin{array}{rccl} \Delta^\bullet_{\mathrm{sub}} : & \Delta & \to & \mathbf{Diff}^r_{\leq 0} \\ & [n] & \mapsto & \Delta^n_{\mathrm{sub}}. \end{array}$$

where Δ^n_{sub} denotes the standard n-simplex with the subspace diffeology furnished by its standard embedding in $\mathbf{R}^{n+1}$.

Proposition 2.10. *The canonical natural transformation* $\operatorname{colim} \circ (\Delta^\bullet_{\mathrm{sub}})^* \to (\pi_{\mathbf{Diff}^r})_!$ *is an equivalence.*

Proof. To see that the image C of $\Delta^\bullet_{\mathrm{sub}}$ satisfies the conditions of Theorem 1.27 we observe that the collection of translations of the standard inclusion $\Delta^d_{\mathrm{sub}} \hookrightarrow \mathbf{A}^d$ form a cover of $\mathbf{A}^d$ ($d \geq 0$), so that we may apply [Lur18, Prop. 20.4.5.1]. The induced functor $\Delta \to C$ is then easily seen to satisfy the conditions of Proposition 1.32, thus verifying the conditions of Theorem 1.27. □

Kihara's simplices It has been a longstanding goal to establish a model structure on diffeological spaces (see e.g. [CW14] and [HS18]). To this end Kihara endows the standard simplices with a new diffeology in [Kih19, § 1.2]. With this diffeology the horn inclusions admit deformation retracts (see Proposition 2.11), allowing Kihara to mimic the construction of the model structure on topological spaces in [Qui67, §II.3], and show that the resulting model category is Quillen equivalent to simplicial sets with the Kan-Quillen model structure. We need Kihara's simplices in order to construct objects satisfying the differentiable Oka principle which is the subject of [Clo24b, §2].

For the convenience of the reader, we repeat the construction of Kihara's simplices: For each $n \geq 1$ and each $0 \leq k \leq n$ we define the set

$$A_k^n := \left\{ (x_0, \dots, x_n) \in \Delta^n \;\middle|\; x_k < 1 \right\}.$$

We now proceed inductively: On Δ^0 and Δ^1 the diffeology is the subspace diffeology coming from $\mathbf{R}^1$ and $\mathbf{R}^2$, respectively. Let $n > 1$, and assume that the diffeologies on the simplices Δ^m for $m < n$ have been defined, then we define a diffeology on A_k^n by exhibiting this set as the underlying set of the quotient

$$\begin{array}{ccc} \Delta^{n-1} \times \{0\} & \hookrightarrow & \Delta^{n-1} \times [0,1) \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ 1 & \hookrightarrow & A_k^n, \end{array}$$

where $\Delta^{n-1} \times [0,1) \to A_n^n$ is given by $(x_0, \dots, x_{n-1}; t) \mapsto ((1-t) \cdot x_0, \dots, (1-t) \cdot x_n, t)$, and similarly for $k \neq n$. Finally, the diffeology on Δ^n is determined by the map $\coprod_{k=0}^n A_k^n \twoheadrightarrow \Delta^n$.

Proposition 2.11 ([Kih19, § 8])**.** *The horn inclusions $\Lambda_k^n \hookrightarrow \Delta^n$ for $n = 2$ and $n \geq k \geq 0$ admit a deformation retract.* □

Definition 2.12. We write

$$\begin{array}{rccc} \Delta^\bullet : & \Delta & \to & \mathbf{Diff}^r \\ & [n] & \mapsto & \Delta^n \end{array}$$

for the cosimplicial object sending each simplex Δ^n to the standard n-simplex endowed with the diffeology described above. The spaces Δ^n for $n \geq 0$ are referred to as ***Kihara's simplices***. Moreover, we write

$$\begin{array}{rcll} \partial\Delta^n & := & \Delta^\bullet_! \partial\Delta^n, & n \geq 0 \\ \Lambda_k^n & := & \Delta^\bullet_! \Lambda_k^n, & n \geq 1,\ n \geq k \geq 0 \end{array}$$

⌟

The proof of the following proposition is completely analogous to the proof of Proposition 2.10.

Proposition 2.13. *The canonical natural transformation* $\operatorname{colim} \circ (\Delta^\bullet)^* \to (\pi_{\mathbf{Diff}^r})_!$ *is an equivalence.* □

Extended cubes See [Clo24b, App. A] or [Cis06, §8.4] for some background on the cube category.

Definition 2.14. We write

$$\begin{array}{rccc} \blacksquare^\bullet : & \square & \to & \mathbf{Diff}^r \\ & \square^n & \mapsto & \mathbf{R}^n \end{array}$$

for the unique symmetric monoidal functor determined by sending the morphisms $\delta^\xi : \square^0 \to \square^1$ to $0 \mapsto \xi$ for $\xi = 0, 1$ (see [Cis06, Prop. 8.4.6]). The spaces $\boldsymbol{\square}^n$ are referred to as the ***extended*** n***-cubes***. ⌟

Proposition 2.15. *The canonical natural transformation* $\operatorname{colim} \circ (\boldsymbol{\square}^\bullet)^* \to (\pi_{\mathbf{Diff}^r})_!$ *is an equivalence.*

Proof. The image of $\boldsymbol{\square}^\bullet$ is given by $\mathbf{Cart}^r$, and the induced functor $\square \to \mathbf{Cart}^r$ is easily seen to satisfy the conditions Proposition 1.33, thus verifying the conditions of Theorem 1.27. □

Closed cubes

Definition 2.16. We write

$$\begin{array}{rccc} \square^\bullet : & \square & \to & \mathbf{Diff}^r \\ & \square^n & \mapsto & [0,1]^n \end{array}$$

for the unique symmetric monoidal functor determined by sending the morphisms $\delta^\xi : \square^0 \to \square^1$ to $0 \mapsto \xi$ for $\xi = 0, 1$ (see [Cis06, Prop. 8.4.6]). The spaces $\square^n$ for $n \geq 0$ are referred to as the ***closed*** n***-cubes***. ⌟

The following proposition may be proved using an obvious adaption of the proofs of Propositions 2.15 & 2.10.

Proposition 2.17. *The canonical natural transformation* $\operatorname{colim} \circ (\boldsymbol{\square}^\bullet)^* \to (\pi_{\mathbf{Diff}^r})_!$ *is an equivalence.* □

2.2.2 Change of regularity

Theorem 2.18. *Let $r \geq s \geq 0$, and denote by $u : \mathbf{Cart}^r \to \mathbf{Cart}^s$ the forgetful functor, then the adjunction $u_* : [(\mathbf{Cart}^r)^{\mathrm{op}}, \mathcal{S}] \underset{\longrightarrow}{\overset{\longleftarrow}{\bot}} [(\mathbf{Cart}^s)^{\mathrm{op}}, \mathcal{S}] : u^*$ restricts to an essential geometric morphism $u_* : \mathbf{Diff}^r \underset{\longrightarrow}{\overset{\longleftarrow}{\bot}} \mathbf{Diff}^s : u^*$, such that $u_! : \mathbf{Diff}^r \to \mathbf{Diff}^s$ sends any r-times differentiable manifold to its underlying s-times differentiable manifold.*

Setting $s = 0$ we obtain the following corollary:

Corollary 2.19. *The underlying topological space of any r-times differentiable manifold calculates its shape.* □

Proof of Theorem 2.18. The forgetful functor $u : \mathbf{Mfd}^r \to \mathbf{Mfd}^s$ is clearly cover reflecting, so that u^* preserves sieves, which shows that u_* restricts to a functor $\mathbf{Diff}^r \to \mathbf{Diff}^s$. As u preserves pullbacks along open embeddings, u satisfies condition iii) of [SGA 4_{I}, Prop. III.1.11], so that $u_! : \widehat{\mathbf{Mfd}^r} \to \widehat{\mathbf{Mfd}^s}$ preserves covering sieves, so that u^* restricts to a functor $\mathbf{Diff}^r \leftarrow \mathbf{Diff}^s$. The functor $u_! : \mathbf{Diff}^r \to \mathbf{Diff}^s$ is obtained by composing the restriction of $u_! : [(\mathbf{Mfd}^r)^{\mathrm{op}}, \mathcal{S}] \to [(\mathbf{Mfd}^s)^{\mathrm{op}}, \mathcal{S}]$ to $\mathbf{Diff}^r$ with the sheafification functor $[(\mathbf{Mfd}^s)^{\mathrm{op}}, \mathcal{S}] \to \mathbf{Diff}^s$. □

2.3 Applications

We now present two applications of the technology developed so far. In §2.3.1 we show that once we decompose $\mathbf{Diff}^r_{\text{ét}}$ into the coproduct (in $\mathbf{Top}$) of the ∞-toposes $\mathbf{Diff}^r_{\text{ét},d}$ of d-dimensional étale differentiable stacks, and moreover have Carchedi's result that the d-th Haefliger stack $\mathbf{H}^d$ is final in $\mathbf{Diff}^r_{\text{ét},d}$ (see Theorem 2.22), then the calculation of the shape of $\mathbf{H}^d$ (as an object $\mathbf{Diff}^r$) follows formally from the way in which $\mathbf{Diff}^r$ is a locally contractible ∞-topos compatibly with its structure as a fractured ∞-topos. Then, in §2.3.2 we observe that the shape of the sheaf on $\mathbf{Diff}^0$ represented by a topological space calculates its singular homotopy type, and thus we are able to harness the descent properties of $\mathbf{Diff}^0$ to prove descent theorems in algebraic topology. We recover Dugger and Isaksen's hypercovering theorem (see [DI04, Th. 1.1], Theorem 2.35) essentially for free, and using simple arguments we obtain Lurie's Seifert-Van Kampen theorem (see [Lur17, Th. A.3.1], Theorem 2.29) as well as the folk theorem that the base space of any principal bundle is a homotopy quotient (see Theorem 2.42).

2.3.1 The shape of the Haefliger stack

The underlying topological groupoid of Γ^d, defined below, now known as the Haefliger groupoid, was introduced by Haefliger in [Hae58] with a view towards applications to the study of foliations. Its classifying space (in the sense of [Seg68]) was first determined in [Seg78, Prop. 1.3], and later Carchedi provided a new calculation of this classifying space in [Car16, Th. 3.7]. The proof we present here is essentially Carchedi's, the only difference being that we may exhibit every step of the proof as a formal manipulation in the calculus afforded by a more systematic account of the theory of locally contractible ∞-toposes and their interactions with fractured ∞-toposes. (Incidentally, similar interactions between shapes and fractured ∞-toposes – although not in this language – are explored by Carchedi in a subsequent article, [Car21], where GAGA like theorems are established for profinite shapes.)

Before turning to the Haefliger stack we quickly explain how to decompose $\mathbf{Diff}^r_{\text{ét},d}$ into a product of ∞-toposes. Denote by $\mathbf{Cart}^r_{\text{ét},d}$ the category of d-dimensional r-times differentiable Cartesian spaces, and by $\mathbf{Diff}^r_{\text{ét},d}$, the ∞-topos of $\mathcal{S}$-valued sheaves on $\mathbf{Cart}^r_{\text{ét},d}$ – the ***d-dimensional étale r-times differentiable stacks***. Observe that $\mathbf{Cart}^r_{\text{ét},d}$ is equivalent to the monoid (viewed as a category) of r-times differentiable embeddings $\mathrm{Emb}^r(\mathbf{R}^d, \mathbf{R}^d)$.

Proposition 2.20. *Let $\{\mathcal{E}_i\}_{i\in I}$ be a family of ∞-toposes indexed by a (small) set I.*

(1) *The coproduct of $\{\mathcal{E}_i\}_{i\in I}$ in* **Top** *is given by the product of $\{\mathcal{E}_i\}_{i\in I}$ in the ∞-categories of ∞-categories.*

(2) *The structure geometric morphism $\iota_i : \mathcal{E}_i = \mathcal{E}_i \times \mathbf{1}_{\mathbf{Cat}} \to \mathcal{E}_i \times \prod_{i\neq j} \mathcal{E}_j = \prod_{i\in I} \mathcal{E}_i$ is essential for every $i \in I$.*

(3) *For any $i \in I$ and any object X in $\mathcal{E}_i$ the geometric morphism $\iota_i : (\mathcal{E}_i)_{/X} \to (\prod_{i\in I} \mathcal{E}_i)_{/(\iota_i)_! X}$ is an equivalence.*

(4) *For any sequence of objects $(X_i)_{i\in I} \in \prod_{i\in I} \mathcal{E}_i$ we have $(X_i)_{i\in I} = \coprod_{i\in I} (\iota_i)_! X_i$.*

(5) *Let $\{C_i\}_{i\in I}$ be a family of small ∞-categories, then*

(5.1) *the equivalence $\prod_{i\in I}[C_i^{\mathrm{op}}, \mathcal{S}] = [\coprod_{i\in I} C_i, \mathcal{S}]$ establishes a bijection*

$$\prod_{i\in I}\Big\{\textit{Grothendieck topologies on } C_i\Big\} = \Big\{\textit{Grothendieck topologies on } \coprod_{i\in I} C_i\Big\};$$

(5.2) *if $(\tau_i)_{i\in I}$ and τ are a pair of corresponding elements under the above bijection, then the functors $C_i \to \coprod_{i\in I} C_i$ both preserve and reflect covering sieves and the induced essential geometric* *morphisms* $\mathbf{Sh}_{C_i,\tau_i} \to \mathbf{Sh}_{(\coprod_{i\in I} C_i),\tau}$ *exhibit* $\mathbf{Sh}_{(\coprod_{i\in I} C_i),\tau}$ *as the coproduct of* $\{\mathbf{Sh}_{C_i,\tau_i}\}_{i\in I}$ *(in* **Top***).*

$\square$

We defer the proof of the above proposition to the end of this subsection. We obtain the following corollary:

Proposition 2.21. *The inclusions* $\mathbf{Cart}^r_{\text{ét},d} \hookrightarrow \mathbf{Cart}^r_{\text{ét}}$ *induce essential geometric morphisms* $\mathbf{Diff}^r_{\text{ét},d} \to \mathbf{Diff}^d_{\text{ét},r}$ *exhibiting* $\mathbf{Diff}^d_{\text{ét},r}$ *as the coproduct of* $\{\mathbf{Diff}^r_{\text{ét},d}\}_{d\geq 0}$*.* $\square$

Now, consider the set-valued presheaf on the topological space $\mathbf{R}^d$ ($d \geq 0$) given by sending U to the set of r-times differentiable embeddings of U into $\mathbf{R}^d$. The étalé space of this presheaf, denoted by Γ^d_0, may naturally be viewed as an object of $\mathbf{Diff}^r_{\text{ét}}$. Its underlying set consists of pairs (x, φ) consisting of a point $x \in \mathbf{R}^d$ together with the germ of an embedding of a neighbourhood of x into $\mathbf{R}^d$. Apart from the structure map $\Gamma^d_0 \to \mathbf{R}^d$ there exists a second étale map $\Gamma^d_0 \to \mathbf{R}^d$ given by sending any element (x, φ) of Γ^d_0 to $\varphi(x)$. Composition of germs endows the simplicial diagram $[n] \mapsto \Gamma^d_n := \Gamma^d_0 \times_{\mathbf{R}^d} \overset{(n+1)\times}{\cdots} \times_{\mathbf{R}^d} \Gamma^d_0$ with the structure of a groupoid object in $\mathbf{Diff}^r_{\text{ét}}$, called the ***Haefliger groupoid***, and is denoted by Γ^d. The ***d-th Haefliger stack***, denoted by $\mathbf{H}^d$, is then the étale differentiable stack presented by Γ^d. As usual, we will identity Γ^d and $\mathbf{H}^d$ with their images under $j_! : \mathbf{Diff}^r_{\text{ét}} \to \mathbf{Diff}^r$. The key to calculating the shape of the Haefliger stack is the following observation by Carchedi:

Theorem 2.22 ([Car19, Th. 3.3])**.** *The d-th Haefliger stack* $\mathbf{H}^d$ *is the final object in* $\mathbf{Diff}^r_{\text{ét},d}$*.*

Sketch of proof. It is enough to show that $\mathbf{Diff}^r_{\text{ét},d}(\mathbf{R}^d, \mathbf{H}^d)$ is contractible. It is nonempty as it contains at least one element obtained by composing the identity map $\mathbf{R}^d \to \Gamma^d_0$ with the cover $\Gamma^d_0 \to \mathbf{H}^d$. Let $f : \mathbf{R}^d \to \mathbf{H}^d$ be a map, then every point $\mathbf{R}^d$ admits a neighbourhood U and a lift

$$\begin{array}{ccc} U & \dashrightarrow & \Gamma^d_0 \\ \big\downarrow & & \big\downarrow \\ \mathbf{R}^d & \longrightarrow & \mathbf{H}^d. \end{array}$$

Choosing U sufficiently small, we may assume that $U \dashrightarrow \Gamma^d_0$ is an embedding, and there exists a diffeomorphism between U and its image in Γ^d_0, corresponding to a lift

$$\begin{array}{ccc} & & \Gamma^d_1 \\ & \nearrow & \big\downarrow \\ U & \hookrightarrow & \Gamma^d_0 \end{array}$$

so that $U \hookrightarrow \Gamma^d_0$ is equivalent to the standard inclusion. Performing this procedure for every point in $\mathbf{R}^d$, we see that f may be represented by the identity map $\mathbf{R}^d \to \Gamma^d_0$. Finally, note that the only automorphism of the identity map $\mathbf{R}^d \to \Gamma^d_0$ in the groupoid $\mathbf{Diff}^r_{\text{ét},d}(\mathbf{R}^d, \mathbf{H}^d)$ is the identity. $\square$

Applying Proposition 2.20.(4) we obtain the following corollary:

Corollary 2.23. *The final object of* $\mathbf{Diff}^r_{\text{ét}}$ *is given by* $\coprod_d \mathbf{H}^d$*.* $\square$

We now calculate the shape of the d-th Haefliger stack ($d \geq 0$).

Theorem 2.24 ([Seg78, Prop. 1.3] & [Car16, Th. 3.7])**.** *For all $d \geq 0$:*

$$(\pi_{\mathbf{Diff}^r})_! \mathbf{H}^d = B\,\mathrm{Emb}(\mathbf{R}^d, \mathbf{R}^d).$$

Proof. We have

$$\begin{aligned}
(\pi_{\mathbf{Diff}^r})_! \mathbf{H}^d &= (\pi_{\mathbf{Diff}^r_{\text{ét}}})_! \mathbf{H}^d && \text{Th. 1.36} \\
&= (\pi_{\mathbf{Diff}^r_{\text{ét},d}})_! \mathbf{H}^d && \text{Props. 2.20 \& 1.7} \\
&= (\pi_{\mathbf{Diff}^r_{\text{ét},d}})_! \left(\mathbf{1}_{\mathbf{Diff}^r_{\text{ét},d}}\right) && \text{Th. 2.22} \\
&= \operatorname{colim} \mathbf{1}_{[(\mathbf{Cart}^r_{\text{ét},d})^{\mathrm{op}}, \mathcal{S}]} && \text{Prop. 1.26} \\
&= (\mathbf{Cart}^r_{\text{ét},d})_{\simeq} && \text{Ex. 1.23} \\
&= B\,\mathrm{Emb}^r(\mathbf{R}^d, \mathbf{R}^d),
\end{aligned}$$

□

Remark 2.25. In order to obtain Segal's original result ([Seg78, Prop. 1.3]) on the classifying space of the underlying topological groupoid of Γ^d, it is enough to observe that

1. $\mathbf{H}^d$ is given as the colimit of (the simplicial diagram) Γ^d,
2. $u_! : \mathbf{Diff}^r \to \mathbf{Diff}^0$ preserves colimits,
3. applying $u_!$ to Γ^d produces the underlying topological groupoid of Γ^d (Theorem 2.18), and
4. the fat topological realisation calculates homotopy colimits of simplicial diagrams in **TSpc**.

⌟

We conclude this subsection by giving a sketch of Carchedi's proof of Theorem 2.24 in [Car16], before supplying a proof of Proposition 2.20. First, Carchedi constructs the shape functors for $\mathbf{Diff}^r_{\text{ét},d}$ and $\mathbf{Diff}^r$ (without identifying them as such) as follows: Denote by $L : \mathbf{TSpc} \to \mathcal{S}$ the localisation functor, then the sequence of functors

$$\mathbf{Cart}^r_{\text{ét},d} \to \mathbf{Mfd}^r_{\text{ét},d} \to \mathbf{Mfd}^r \to \mathbf{TSpc} \to \mathcal{S}$$

gives rise to the sequence of cocontinuous functors

$$[(\mathbf{Cart}^r_{\text{ét},d})^{\mathrm{op}}, \mathcal{S}] \to \mathbf{Diff}^r_{\text{ét},d} \to \mathbf{Sh}_{\mathbf{Mfd}^r_{\text{ét},d}} (= \mathbf{Diff}^r_{\text{ét},d}) \to \mathbf{Diff}^r \to \mathcal{S}, \tag{11}$$

as the composition $\mathbf{Mfd}^r \to \mathbf{TSpc} \to \mathcal{S}$ preserves colimits of hypercovers by Theorem 2.35 (and the fact that fat topological realisations are homotopy colimits), and because the functor $\mathbf{Mfd}^r_{\text{ét},d} \to \mathbf{Mfd}^r$ preserves covering sieves by [SGA 4_{I}, Prop. III.1.11]. Then, one observes that the composition of all the functors in (11) sends $\mathbf{R}^d$ to $\mathbf{1}_{\mathcal{S}}$ for all $d \geq 0$, so that by cocontinuity the composition is simply given by the colimit functor. Thus the shape of the d-th Haefliger stack ($d \geq 0$) is again given by $B\,\mathrm{Emb}^r(\mathbf{R}^d, \mathbf{R}^d)$. To obtain the comparison with Segal's result (as in Remark 2.25) it is enough to observe that the shape of the colimit of any simplicial diagram of (not necessarily 2^{nd}-countable, Hausdorff) manifolds is equivalent to the homotopy type of the fat topological realisation of the underlying simplicial diagram of topological spaces, again by Theorem 2.35 and the fact that fat topological realisations are homotopy colimits.

Proof of Proposition 2.20.

(1) This is Proposition [Lur09, 6.3.2.1].

(2) The initial topos is given by $\mathbf{1}_{\mathbf{Cat}}$, and for any ∞-topos $\mathcal{E}$ the unique geometric morphism $\varnothing : \mathbf{1}_{\mathbf{Cat}} \to \mathcal{E}$ is essential, where the left adjoint to the pullback functor is given by sending the unique object of $\mathbf{1}_{\mathbf{Cat}}$ to the initial object of $\mathcal{E}$.

(3) The functor $\mathbf{Cat}_{\mathbf{1}/} \to \mathbf{Cat}$ taking any pointed ∞-category $\mathbf{1} \xrightarrow{c} C$ to $C_{/c}$ is right adjoint to the cone functor, and thus preserves limits. The ∞-category $(\{\mathcal{E}_i\}_{i\in I})_{/(\iota_i)_!X}$ is obtained by taking the product of $(\prod_{i\neq j} \mathcal{E}_i)_{/\varnothing_!\mathbf{1}} = (\prod_{i\neq j} \mathcal{E}_i)_{/\varnothing} = \mathbf{1}$ and $(\mathcal{E}_i)_{/X}$.

(4) For any object $Y \in \prod_{i\in I} \mathcal{E}$ we have

$$\begin{aligned} \left(\prod_{i\in I} \mathcal{E}_i\right) \left(\coprod_{i\in I} (\iota_i)_! X_i, Y\right) &= \prod_{i\in I} \left(\prod_{i\in I} \mathcal{E}_i\right) ((\iota_i)_! X_i, Y) \\ &= \prod_{i\in I} \left(\prod_{i\in I} \mathcal{E}_i\right) (X_i, \iota_i^* Y) \\ &= \left(\prod_{i\in I} \mathcal{E}_i\right) ((X_i)_{i\in I}, Y) \end{aligned}$$

where the last isomorphism follows from [Lur09, Lm. 6.3.3.6].

(5)

(5.1) This is an immediate consequence of statement (3) of the theorem.

(5.2) That the functors $C_i \to \coprod_{i\in I} C_i$ both preserve and reflect covering sieves again follows from (3). The equivalence $\prod_{i\in I}[C_i^{\mathrm{op}}, \mathcal{S}] \xleftarrow{\simeq} [\coprod_{i\in I} C_i, \mathcal{S}]$ restricts to a fully faithful functor $\mathbf{Sh}_{C_i,\tau_i} \hookleftarrow \mathbf{Sh}_{(\coprod_{i\in I} C_i),\tau}$. It remains to show that any presheaf on $\coprod_{i\in I} C_i$ which is sent to an object in $\mathbf{Sh}_{C_i,\tau_i}$ lies in $\mathbf{Sh}_{(\coprod_{i\in I} C_i),\tau}$.

□

2.3.2 Algebraic topology and descent

Let X be a topological space covered by two open sets U and V such that $X, U, V, U \cup V$ are connected, then the Seifert – Van Kampen theorem states that the square

$$\begin{array}{ccc} \pi_1 X & \longleftarrow & \pi_1 V \\ \uparrow & & \uparrow \\ \pi_1 U & \longleftarrow & \pi_1(U \cap V) \end{array}$$

is a pushout (for any basepoint in $U \cap V$). In fact, more is true: Let $L : \mathbf{TSpc} \to \mathcal{S}$ be the localisation functor along the Serre-Quillen weak equivalences, then the pushout square in $\mathbf{TSpc}$

$$\begin{array}{ccc} X & \longleftarrow & V \\ \uparrow & & \uparrow \\ U & \longleftarrow & U \cap V \end{array} \tag{12}$$

is carried by L to a pushout square in $\mathcal{S}$, i.e., (12) is a homotopy pushout. Squares such as (12) encode glueing data for topological spaces, so that the Seifert – Van Kampen theorem reflects how descent for topological spaces interacts with their singular homotopy types.

We give a quick proof of the statement that (12) is a homotopy pushout, which will function as a paradigm for our new proof of Theorem 2.29 (Lurie's Seifert – Van Kampen theorem), as well as the proofs of Theorem 2.35 (Dugger and Isaksen's hypercovering theorem) and Theorem 2.42 (which states that the base space of any principal bundles is a homotopy quotient). Denote by $v : \mathbf{Cart}^0 \hookrightarrow \mathbf{TSpc}$ the inclusion of the category of Cartesian spaces into the category of topological spaces, then $v_! : [(\mathbf{Cart}^r)^{\mathrm{op}}, \mathcal{S}] \to \mathbf{TSpc}$ sends sieves generated by covers consisting of jointly surjective open embeddings to isomorphisms, so that we obtain an adjunction

$$v_! : \mathbf{Diff}^0 \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{TSpc} : v^* \ .$$

Proposition 2.26. *There exists a canonical natural equivalence:*

$$\begin{array}{ccc} \mathbf{Diff}^0_{\leq 0} & \xleftarrow{\ v^*\ } & \mathbf{TSpc} \\ & \overset{\sim}{\Longrightarrow} & \\ {\scriptstyle \pi_!} \searrow & & \swarrow \\ & \mathcal{S} & \end{array}$$

In other words, for any topological space X, the shape of v^*X is canonically equivalent to its singular homotopy type.

Proof. Observe that the standard simplex functor

$$\Delta \to \mathbf{TSpc}$$

factors as $\Delta \xrightarrow{\Delta_!} \mathbf{Diff}^0 \xrightarrow{v_!} \mathbf{TSpc}$. Then, by Proposition 2.10 we obtain the diagram

$$\begin{array}{ccccc} \widehat{\Delta} & \xleftarrow{\ \Delta^*\ } & \mathbf{Diff}^0_{\leq 0} & \xleftarrow{\ v^*\ } & \mathbf{TSpc} \\ & \overset{\sim}{\Longrightarrow} & \big\downarrow {\scriptstyle \pi_!} & & \\ & {\scriptstyle \pi_!} \searrow & \mathcal{S} & & \end{array}$$

The desired natural equivalence is then obtained by whiskering. ☐

Warning 2.27. Proposition 2.26 does not imply that the singular homotopy type of a topological space coincides with its shape. For example, the shape of the Hawaiian earring is not even representable. ⌟

Remark 2.28. Proposition 2.26 and the attendant Theorems 2.29, 2.35, 2.42 remain true when we replace $\mathbf{Diff}^0$ with $\mathbf{Diff}^r$ for $r > 0$, but we find this circumstance bewildering, so we have opted to fix $r = 0$ until the end of this chapter. ⌟

Now, observe that the commutative square

$$\begin{array}{ccc} v^*X & \longleftarrow & v^*V \\ \uparrow & & \uparrow \\ v^*U & \longleftarrow & v^*(U \cap V) \end{array}$$

is a pushout square in $\mathbf{Diff}^r$ (which can be seen, e.g., by pulling back along all continuous maps $\mathbf{R}^d \to v^*X$ $(d \geq 0)$), so that (12) is a homotopy pushout square by Proposition 2.26 and the fact that $\pi_!$ preserves colimits.

Lurie's Seifert – Van Kampen theorem We now prove Lurie's far reaching generalisation of the Seifert – Van Kampen Theorem:

Theorem 2.29 ([Lur17, A.3.1]). *Let X be a topological space, and denote by $\mathbf{Open}_X$ the category of open subsets of X (ordered by inclusion). Furthermore, let A be a small category, and $\chi : A \to \mathbf{Open}_X$, a functor. For each element $x \in X$ denote by A_x the full subcategory of A spanned by those objects $a \in A$ such that $x \in \chi(a)$. If $(A_x)_{\simeq} = \mathbf{1}_{\mathcal{S}}$ for each $x \in X$, then the cocone $A^{\triangleright} \to \mathbf{TSpc}$ obtained by composing the unique cocone $A^{\triangleright} \to \mathbf{Open}_X$ with apex X on χ with the functor $\mathbf{Open}_X \to \mathbf{TSpc}$ is a homotopy colimit.* □

The version of the Seifert – Van Kampen Theorem presented above is then obtained by setting $A = U \leftarrow U \cap V \to V$, and letting χ be the inclusion $A \hookrightarrow \mathbf{Open}_X$.

Proof of Theorem 2.29. The composition of $\mathbf{Diff}^0 \xleftarrow{u^*} \mathbf{TSpc} \leftarrow \mathbf{Open}_X$ sends any covering $\{U \subseteq V\}$ to a covering in $\mathbf{Diff}^0$ (as can be seen by pulling back the inclusions $u^*U \hookrightarrow u^*V$ along all maps $\mathbf{R}^d \to u^*V$), and moreover preserves finite limits, yielding a geometric morphism $(u_X)_* : \mathbf{Diff}^0 \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathbf{Sh}_X : u_X^*$. We must show that $A^{\triangleright} \to \mathbf{Open}_X \to \mathbf{Sh}_X \xrightarrow{u_X^*} \mathbf{Diff}^0 \xrightarrow{\pi_!} \mathcal{S}$ is a colimit. As $\mathbf{Diff}^0$ is hypercomplete and u_X^* preserves ∞-connective morphisms, it is enough to show that $\operatorname{colim} \chi \to X$ is ∞-connected (in $\mathbf{Sh}_X$), which can be checked by showing that it is sent to an isomorphism by the stalk $x^* : \mathbf{Sh}_X \to \mathcal{S}$ for every elements $x \in X$ by [Lur17, Lm. A.3.9.]. The left Kan extension of the constant functor $\mathbf{1}_{\mathcal{S}} : A_x \to \mathcal{S}$ is given by $A \xrightarrow{\chi} \mathbf{Sh}_X \xrightarrow{x^*} \mathcal{S}$, so that we obtain

$$\mathbf{1}_{\mathcal{S}} = \operatorname{colim}(\mathbf{1}_{\mathcal{S}} : A_x \to \mathcal{S}) = \operatorname{colim} x^*\chi = x^* \operatorname{colim} \chi \to x^*X = \mathbf{1}_{\mathcal{S}}$$

where the first isomorphism holds by assumption. □

Corollary 2.30. *Let X be a topological space, and $R \hookrightarrow X$ a covering sieve (in $\widehat{\mathbf{Open}_X}$), then the cocone $R^{\triangleright} \to \mathbf{TSpc}$ obtained by composing the colimiting cocone $R^{\triangleright} \to \mathbf{Open}_X$ with $\mathbf{Open}_X \to \mathbf{TSpc}$ is a homotopy colimit.*

Proof. Set $A = R$, and χ equal to the inclusion $R \hookrightarrow \mathbf{Open}_X$. For every point $x \in X$ the category A_x is filtered, and thus its classifying space is contractible. □

In [Lur17] Lurie first gives a technical proof of Corollary 2.30, from which he derives Theorem 2.29 using arguments similar to those used in the proof of Theorem 2.29.

R-epimorphisms The two remaining theorems of this section, Theorem 2.35 and Theorem 2.42, are most naturally expressed using **R**-*epimorphisms*, a common generalisation of locally trivial bundles and open covers.

Proposition 2.31. *Let $X \to Y$ be a continuous map, then the following are equivalent:*

(1) *The map* $u^*X \to u^*Y$ *in* $\mathbf{Diff}^0$ *is an effective epimorphism.*

(2) *For every* $d \geq 0$*, every continuous map* $\mathbf{R}^d \to Y$*, and every point* $x \in \mathbf{R}^d$ *there exists a neighbourhood* U *of* x *and a lift*

$$\begin{array}{ccccc} & & & & X \\ & & & \nearrow & \downarrow \\ U & \hookrightarrow & \mathbf{R}^n & \longrightarrow & Y \end{array}$$

$\square$

Definition 2.32. A continuous map $X \to Y$ is an ***R-epimorphism*** if it satisfies the equivalent conditions of Proposition 2.31. ⌟

Dugger and Isaksen's hypercovering theorem

Definition 2.33. Let X be a topological space, then a simplicial diagram $U : \Delta^{\text{op}} \to \mathbf{TSpc}_{/X}$ is an ***R-hypercover*** if $U^{\Delta^n} \to U^{\partial\Delta^n}$ is an $\mathbf{R}$-epimorphism for all $n \geq 0$. ⌟

Example 2.34. Any ordinary hypercover of a topological space is an $\mathbf{R}$-hypercover. ⌟

Theorem 2.35 ([DI04, Th. 1.1])**.** *Let* X *be a topological space, and* $U : \Delta^{\text{op}} \to \mathbf{TSpc}_{/X}$*, an* $\mathbf{R}$*-hypercover, then the corresponding cocone* $\overline{U} : (\Delta^{\text{op}})^{\triangleright} \to \mathbf{TSpc}$ *is a homotopy colimit.*

Proof. The functor $\mathbf{Diff}^0 \leftarrow \mathbf{TSpc} : v^*$ preserves limits, and sends $\mathbf{R}$-epimorphism to effective epimorphisms by definition. Therefore, the composition of $(v^*_{/X}U)^{\partial\Delta^n} \xrightarrow{\simeq} v^*(U^{\partial\Delta^n}) \to v^*U^{\Delta^n}$ is an effective epimorphism for every $n \geq 1$, so that $v^*_{/X}U$ is a hypercover. Thus, $v^*\overline{U}$ is a colimit by descent, and we may apply Proposition 2.26. $\square$

Principal bundles Until the end of this section G denotes a topological group. Assume that G acts on a topological space X. If the action is principal, then it is often taken for granted that X/G is homotopically well-behaved. For an example of what what is meant by this: if in addition to being principal, X is moreover contractible, then X/G is a model for BG. To obtain a precise notion of this homotopical well-behavedness, we note that the localisation functor $L : \mathbf{TSpc} \to \mathcal{S}$ commutes with finite products, so that we obtain an action of LG on LX. We then say that X/G is a ***homotopy quotient*** of the action of G on X if the comparison map $LX/LG \to L(X/G)$ is an isomorphism. We will prove in Theorem 2.42 that if the action of G on X is principal, then X/G is indeed a homotopy quotient.

Remark 2.36. It is often claimed, erroneously, that X/G is a homotopy quotient for any *free* action. To see that this is not the case, let G act on a copy of itself equipped with the trivial topology, then the quotient is a point. If the quotient were a homotopy quotient, it would have to model the classifying space of G, which is only true if G itself is weakly contractible. It is true however, that any free quotient in any strict test topos $\mathcal{E}$ is a homotopy quotient, as quotients by free actions commute with the inclusion of $\mathcal{E}$ into its associated hypercomplete ∞-topos. ⌟

Our notion of *homotopy quotient* agrees with more traditional notions of homotopically well-behaved quotients. For example, the category of topological spaces with a continuous G-action, $\mathbf{TSpc}_G$, admits a model structure in which the weak equivalences are those equivariant maps whose underlying maps of topological spaces are weak equivalences (see [M+96, Th. VI.5.2]), and one may ask when X/G has the same weak homotopy type as $X//G$, where $\_//G$ denotes the derived functor of the quotient functor $\mathbf{TSpc}_G \to \mathbf{TSpc}$. These two notions agree by the following proposition.

Proposition 2.37. *The functor* $\mathbf{TSpc}_G \to \mathcal{S}_{LG}$ *is a localisation along the weak equivalences in* $\mathbf{TSpc}_G$.

Proof. The functor $\mathcal{S}_{LG} \leftarrow \mathbf{TSpc}_G$ factors as $\mathcal{S}_{LG} \leftarrow (\widehat{\Delta})_{sG} \leftarrow \mathbf{TSpc}_G$ (where sG is the total singular complex of G), so the proposition follows from [Clo24b, Th. 1.4.14] and the fact that $(\widehat{\Delta})_{sG} \leftarrow \mathbf{TSpc}_G$ preserves weak equivalences and induces and equivalence of ∞-categories upon localisation (see [DK84, 1.7]). $\square$

Using classical methods we are only aware of a proof of $X//G \sim X/G$ for a principal action under the additional (mild) assumptions that G is well pointed, and X is a compactly generated weakly Hausdorff space: By [Rie14, 9.2.10], $X//G$ may be computed as the topological realisation of $\cdots\ X \times G \times G \Rrightarrow X \times G \rightrightarrows X$, and this topological realisation is weakly equivalent to X/G by [May75, Props. 7.1 & 8.5] (which relies on technical pointset topological arguments).

We will now systematically investigate the relationship between principal actions and homotopy quotients.

Definition 2.38. An $\mathbf{R}$***-principal*** G***-bundle*** is an $\mathbf{R}$-epimorphism $P \to B$ together with a fibre preserving action of G on P, such that the shearing map $P \times G \to P \times_B P$ is a homeomorphism. ⌟

Example 2.39. Any principal G-bundle is an $\mathbf{R}$-principal G-bundle. ⌟

Lemma 2.40. *Let* $P \to B$ *be an* $\mathbf{R}$*-principal* G*-bundle, then the diagram*

$$\begin{array}{ccc} \mathbf{Diff}^0_{v^*G} & \longleftarrow & \mathbf{TSpc}_G \\ {\scriptstyle v^*P\times_{v^*G}(\_)}\downarrow & & \downarrow{\scriptstyle P\times_G(\_)} \\ \mathbf{Diff}^0 & \longleftarrow & \mathbf{TSpc} \end{array}$$

commutes.

Proof. We will show that the natural transformation $v^*P\times_{v^*G}v^*(\_) \to v^*(P\times_G\_)$, obtained by whiskering $P \times \_ : \mathbf{TSpc} \to \mathbf{TSpc}$ with $v^*(\_)/v^*G \to v^*(\_/G)$, is a natural isomorphism.

Pulling back $X \times_G P \to B$ along $P \to B$ yields the Cartesian natural transformation

$$\begin{array}{ccccccc} \cdots\ X \times G \times G \times P & \Rrightarrow & X \times G \times P & \rightrightarrows & X \times P & \longrightarrow & X \times_G P \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \cdots\ P \times_B P \times_B P & \Rrightarrow & P \times_B P & \rightrightarrows & P & \longrightarrow & B \end{array}$$

As v^* preserves limits, we see that

$$\cdots\ v^*X \times v^*G \times v^*G \times v^*P \Rrightarrow v^*X \times v^*G \times v^*P \rightrightarrows v^*X \times v^*P$$

is the Čech complex both of $v^*X \times v^*P \to v^*(X \times_G P)$ and of $v^*X \times v^*P \to v^*X \times_{v^*G} v^*P$, so that the comparison map $v^*X \times_{v^*G} v^*P \to v^*(X \times_G P)$ is an isomorphism by descent. $\square$

Theorem 2.41. *Let $P \to B$ be a an $\mathbf{R}$-principle G-bundle, and X a G-space, then the comparison map $LX \times_{LG} LP \to L(X \times_G P)$ is an isomorphism in $\mathcal{S}$.*

Proof. As $v^*P \times_{v^*G} v^*X = (v^*P \times v^*X)/v^*G$, the theorem follows from Lemma 2.40 and Propositions 1.24 & 2.26. □

Setting $X = \mathbf{1}$ yields the following corollary:

Corollary 2.42. *If $P \to B$ is an $\mathbf{R}$-principal G-bundle, then B is the homotopy quotient of the G-space P.* □

Corollary 2.43. *If $E \to B$ is a $\mathbf{R}$-principal G-bundle with E weakly contractible, the comparison map $B(LG) \to LB$ is an isomorphism in $\mathcal{S}$.* □

Traditionally, the Borel construction is often used as the *definition* of homotopy quotients. We verify that the Borel construction computes homotopy quotients in our sense.

Proposition 2.44. *The functor $\_ \times_G E : \mathbf{TSpc}_G \to \mathbf{TSpc}$ preserves weak equivalences, and the induced functor $\mathcal{S}_{LG} \to \mathcal{S}$ is canonically isomorphic to $\_/LG$.*

Proof. For any topological space X the map $v^*X \times v^*E \to v^*X$ is a shape equivalence, so that the outer square in

$$\begin{array}{ccccc} \mathcal{S}_{LG} & \longleftarrow & \mathbf{Diff}^0_G & \longleftarrow & \mathbf{TSpc}_G \\ \downarrow{\scriptstyle \_/LG} & & \downarrow{\scriptstyle \_/v^*G} & & \downarrow{\scriptstyle E\times_G \_} \\ \mathcal{S} & \longleftarrow & \mathbf{Diff}^0 & \longleftarrow & \mathbf{TSpc} \end{array}$$

commutes by Lemma 2.40 and Proposition 1.24. □

Conventions and notation

- The term ∞-*category* means *quasi-category*.
- We identify ordinary categories with their nerves, and consequently do not notationally distinguish between ordinary categories and their nerves.
- $[\_,\_]$ denotes the internal hom in $\widehat{\Delta}$, the category of simplicial sets.
- Let C, D be ∞-categories, and $W \subseteq C$, a subcategory, then $[C,D]_W$ denotes the subcategory of $[C,D]$ spanned by those functors sending every morphism in W to an isomorphism.
- Let X be a simplicial set, then $X_{\simeq}$ denotes the classifying space of X, given e.g. by $\mathrm{Ex}^{\infty}A$.
- ∞-categories (including ordinary categories) are denoted by C, D, ...
- Let C be an ∞-category and let $x, y \in C$ be two objects, then the homotopy type of morphisms from x to y is denoted by $C(x,y)$.
- A final object in an ∞-category C is denoted by $\mathbf{1}_C$, or simply by $\mathbf{1}$, when C is clear from context.

- For any ∞-category C we denote its subcategory of n-truncated objects by $C_{\leq n}$.
- For any ∞-category C with finite products and any group object G in C, we denote C_G the category of G-objects in C.
- For A any small ordinary category $\widehat{A}$ denotes the category of (set-valued) presheaves on A.
- For any two categories C, D, an arrow $C \hookrightarrow D$ denotes a fully faithful functor.
- We use the following notation for various ∞-categories:
  - We adopt the "French" tradition of denoting the ordinary category of presheaves on any small ordinary category A by $\widehat{A}$. E.g., the category of simplicial sets is denoted by $\widehat{\Delta}$.
  - Canonical isomorphisms are often denoted by equality signs. (An isomorphism is canonical if it originates from a universal property. More precisely, let $u : X \to C$ be a right fibration, and x, x' two final objects in X, then for any morphism $x \to x'$ the morphism $ux \to ux'$ is a canonical isomorphism, and we may write $x = x'$.)
  - Δ denotes the category of simplices. Its objects are denoted by Δ^n or $[n]$, depending on context.
  - $\square$ denotes the category of cubes (see [Cis06, Ch. 8]).
  - $\mathcal{S}$ denotes the ∞-categories of homotopy types.
  - **Cat** denotes the ∞-category of ∞-categories.
  - **Top** denotes the ∞-category of ∞-toposes.
  - Let X be a topological space, then $\mathbf{Open}_X$ denotes the locale of open subsets of X.
  - **Set** denotes the category of sets.
  - **TSpc** denotes the category of topological spaces.
  - Δ**TSpc** is the full subcategory of **TSpc** spanned by the Δ-generated topological spaces.
  - $\mathbf{Mfd}^r$ denotes the category of r-times differentiable smooth manifolds and smooth maps.
  - $\mathbf{Cart}^r$ denotes the full subcategory of $\mathbf{Mfd}^r$ spanned by the spaces of $\mathbf{R}^n$ $(0 \leq n < \infty)$.
  - $\mathbf{Diff}^r$ denotes, equivalently, the ∞-category of sheaves on $\mathbf{Mfd}^r$ or $\mathbf{Cart}^r$.
- We denote ∞-toposes by $\mathcal{E}, \mathcal{F}, \ldots$, when they are thought of as ambient settings in which to do geometry, and by $\mathcal{X}, \mathcal{Y}, \ldots$, when they are thought of as geometric objects in their own right.

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